Path Category for Free

Open Morphisms from Coalgebras with Non-Deterministic Branching

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Categorical Approaches to Bisimilarity

Transition type

Functor $F\colon \mathbb{C} \to \mathbb{C}$

Coalgebra

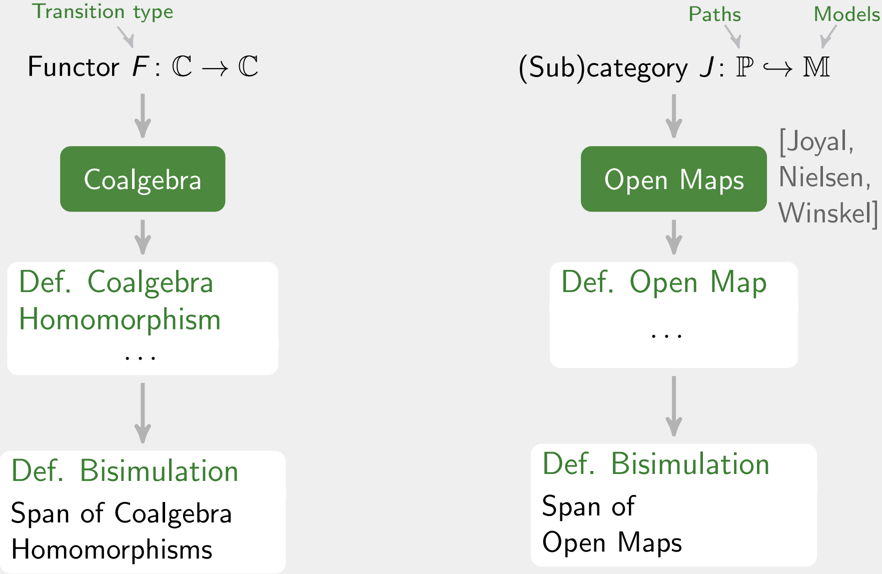
Def. Coalgebra Homomorphism

$\ldots$

Def. Bisimulation

Span of Coalgebra Homomorphisms

Categorical Approaches to Bisimilarity



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Models

(Sub)category $J\colon \mathbb{P} \hookrightarrow \mathbb{M}$

Open Maps

[Joyal, Nielsen, Winskel]

Def. Open Map

…

Def. Bisimulation

Span of Open Maps

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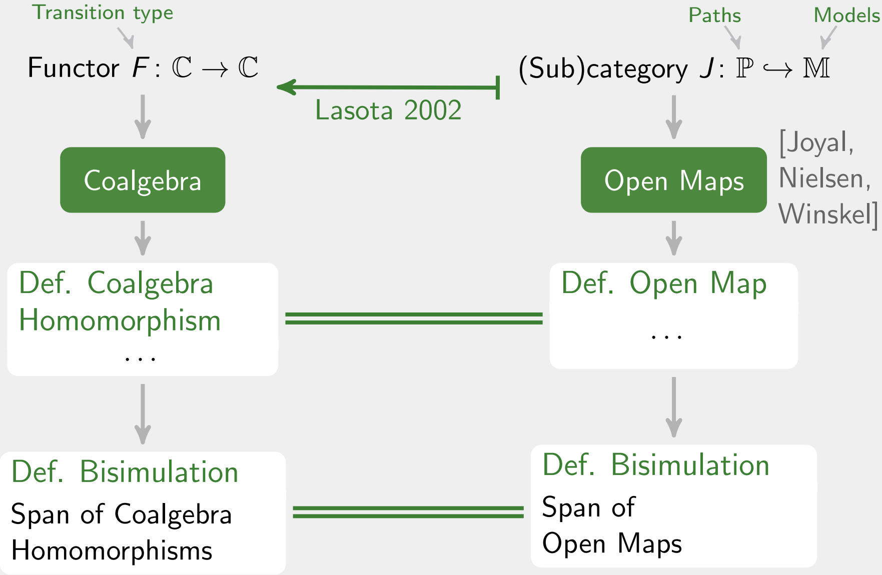

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Lasota 2002

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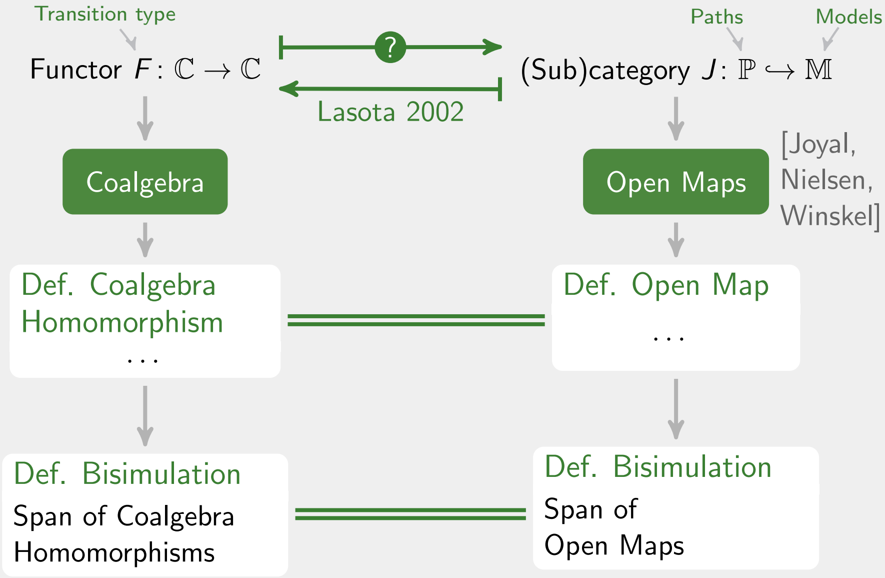

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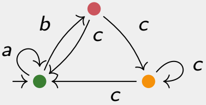
Motivating Example: Category LTS_A

Objects: (X, x_0, Δ)

states X, initial state $x_0 \in X$, transitions $\Delta \subseteq X \times A \times X$.



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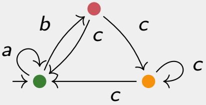

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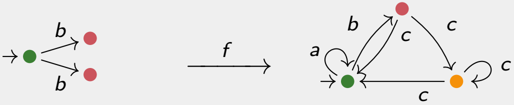

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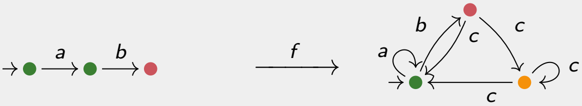

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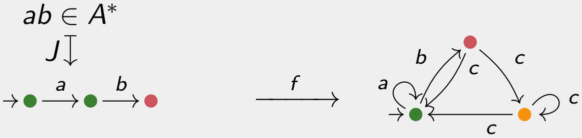

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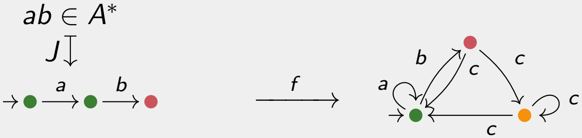

Paths

prefix order

Functor $J\colon (A^*, \leq) \longrightarrow \mathsf{LTS}_A$

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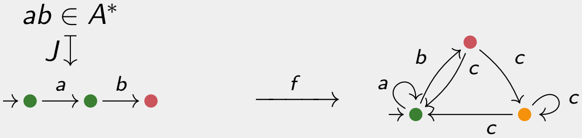

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prefix order

Run of $w \in A^*$ in (X,x_0,Δ)

$f\colon Jw \to (X,x_0,\Delta)$ in LTS_A

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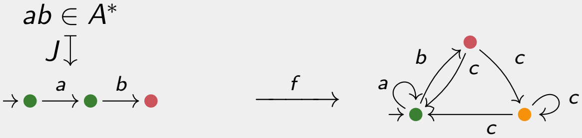

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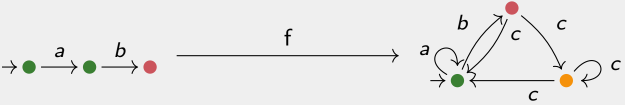
Functor $J\colon \mathbb{P} \longrightarrow \mathbb{M}$

Run of $w \in \mathbb{P}$ in $M \in \mathbb{M}$

$f\colon Jw \to M$ in $\mathbb{M}$

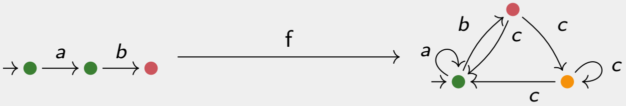
Open Maps



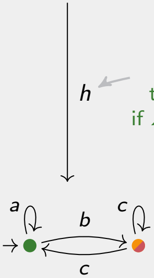
$$Jw \xrightarrow{f} X$$

Joyal, Nielsen, Winskel ’96

Open Maps



$$\begin{tikzcd}
Jw \arrow[r, "f"] & X \arrow[d, "h"] \\
 & Y
\end{tikzcd}$$



reflects
transitions
if X reachable

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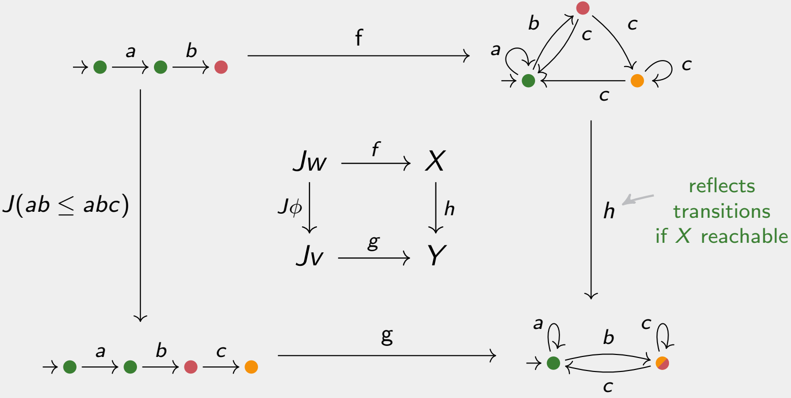

$$\begin{tikzcd}
Jw \arrow[r, "f"] \arrow[d, "J\phi"'] & X \arrow[d, "h"] \\
Jv \arrow[r, "g"] & Y
\end{tikzcd}$$

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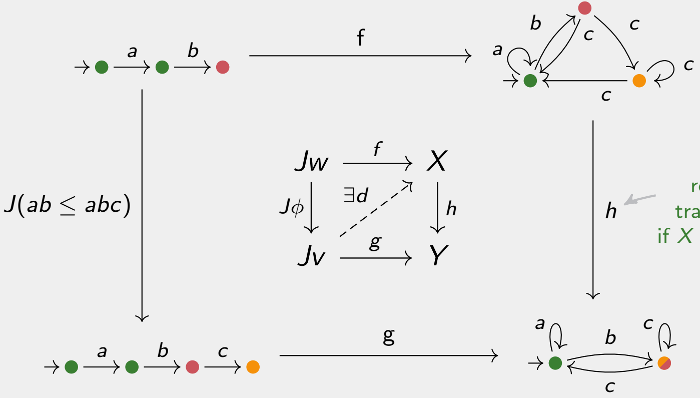
Open Maps



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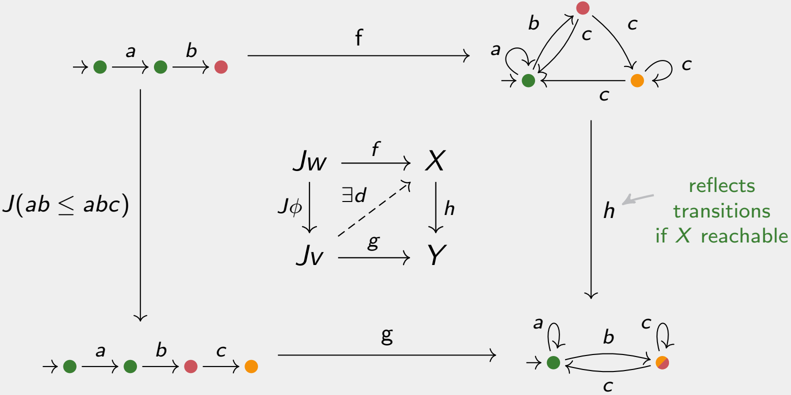

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reflects transitions if X reachable

Definition: h open. . .

. . ., if for every such square, there exists some dialgonal lifting d.

Joyal, Nielsen, Winskel '96

Pointed Coalgebras & Lax homomorphisms

$$\mathsf{LTS}_A \iff \mathsf{LCoalg}(1, \mathcal{P}(A \times (-)))$$

$X, x_0 \in X,$
$\Delta \subseteq X \times A \times X$ $\iff$

1-pointed $\mathcal{P}(A\times(-))$-coalgebra

$$1 \xrightarrow{x_0} X \xrightarrow{\xi} \mathcal{P}(A\times X)$$

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$$\begin{tikzcd}(X, x_0, \Delta) \arrow[d, "h"'] \\ (Y, y_0, \Delta')\end{tikzcd} \iff$$

Lax Coalgebra Homomorphism

Point-wise order $\subseteq$ on $\mathsf{Set}(X, \mathcal{P}Z)$

$$\begin{tikzcd}
1 \arrow[r, "x_0"] \arrow[dr, bend right, "y_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[r, "\xi"] \arrow[d, "h"'] \arrow[dr, phantom, "\rotatebox{-90}{$\subseteq$}"] & \mathcal{P}(A\times X) \arrow[d, "\mathcal{P}(A\times h)"] \\
& Y \arrow[r, "\zeta"'] & \mathcal{P}(A\times Y)
\end{tikzcd}$$

Pointed Coalgebras & Lax homomorphisms

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$h\colon X \to Y$
is open $\overset{}{\underset{X \text{ reachable}}{\iff}}$ h (proper) coalgebra homomorphism
$\zeta \cdot h = \mathcal{P}(A\times h) \cdot \xi$

Pointed Coalgebras & Lax homomorphisms

$\mathsf{LTS}_A \iff \mathsf{LCoalg}(I, T\cdot F)$ e.g. $TX = \mathcal{P}X$, $FX = A \times X$

$X, x_0 \in X,$
$\Delta \subseteq X \times A \times X$ $\iff$

I-pointed $T(F(-))$-coalgebra

$$I \xrightarrow{x_0} X \xrightarrow{\xi} T(FX)$$

$$\begin{tikzcd}
(X, x_0, \Delta) \arrow[d, "h"'] \\
(Y, y_0, \Delta')
\end{tikzcd}$$

$\iff$

Lax Coalgebra Homomorphism

Point-wise order $\subseteq$ on $\mathbb{C}(X, TZ)$

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& Y \arrow[r, "\zeta"'] & T(FY)
\end{tikzcd}$$

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Main Result

Theorem

Branching

Input

Given:

- Functors $T,F\colon \mathbb{C} \to \mathbb{C}$ with order $\subseteq$ on $\mathbb{C}(X, TY)$
- F admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$?
- $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\perp} 1$ plus axioms (T powerset-like) ?

Then there is a path category $\mathrm{Path}(I, F+1)$?
and for every $h\colon X \to Y$ in $\mathrm{LCoalg}(I, TF)$:

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Canonical Trace Semantics: $\mathrm{LCoalg}(I, TF) \to \coprod_{n\geq 0} \mathbb{C}(I, F^n 1)$

$\mathrm{trace}(X) = \{[w] \mid Jw \to X\}$ preserved by coalgebra hom.

Definition: F-precise morphism

$f\colon X \to FY$ is F-**precise** if

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & FW \arrow[d, "Fw"] \\ FY \arrow[r, "Fz"] & FZ \end{tikzcd} \quad \overset{\exists d}{\Longrightarrow} \quad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & FW \\ FY \arrow[ur, "Fd"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} & C \arrow[d, "w"] \\ Y \arrow[ur, "d"] \arrow[r, "z"] & D \end{tikzcd}$$

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Intuition in Sets

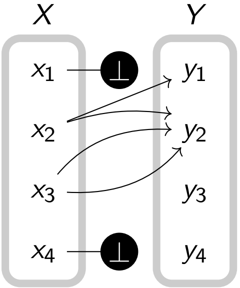
$f\colon X \to FY$
is F-precise

$\iff$

every $y \in Y$ is
mentioned precisely once
in the definition of f

F-precise = every $y \in Y$ is mentioned precisely once

Example: $FY = Y \times Y + \{\bot\}$ and $f\colon X \to FY$



$$f\colon\ X \longrightarrow Y \times Y + \{\bot\}$$

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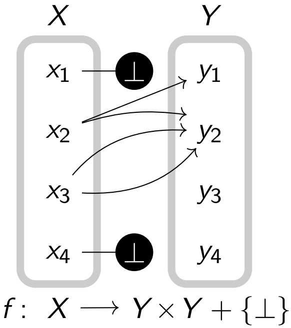
Example: $FY = Y \times Y + \{\perp\}$ and $f : X \to FY$



$$f(x_1) = \perp$$

$$f(x_2) = (y_1, y_2)$$

$$f(x_3) = (y_2, y_2)$$

$$f(x_4) = \perp$$

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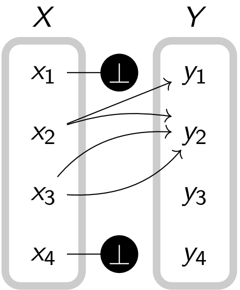

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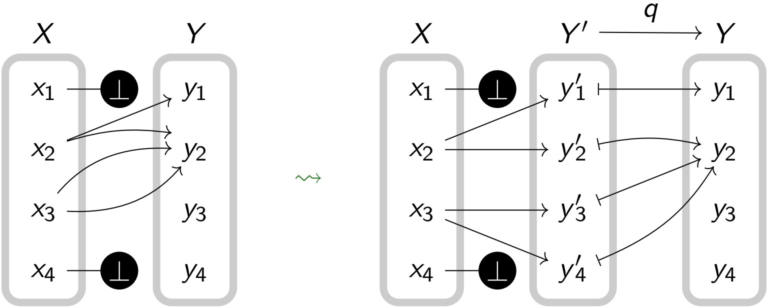

$f\colon\; X \longrightarrow Y \times Y + \{\bot\}$ $\qquad$ $f'\colon\; X \longrightarrow Y' \times Y' + \{\bot\}$

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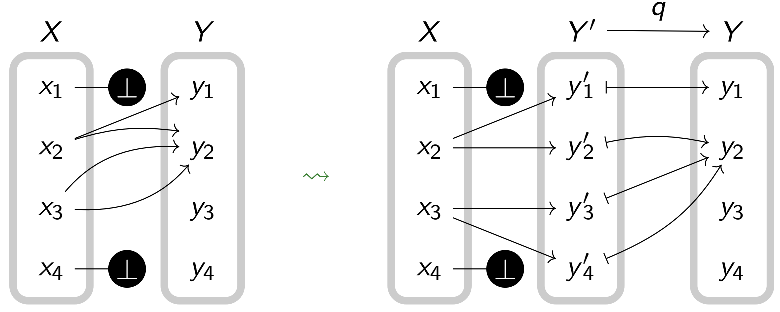

$f\colon\; X \longrightarrow Y \times Y + \{\perp\}$ $\qquad$ $f'\colon\; X \longrightarrow Y' \times Y' + \{\perp\}$

Def.: $F\colon \mathbb{C} \to \mathbb{C}$ admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$

$\forall f$, $X \in \mathcal{S}$,
$\exists Y' \in \mathcal{S}$, f' precise:

$$\begin{tikzcd}
X \arrow[r, dashed, "\exists f'"] \arrow[dr, "\forall f"'] & FY' \arrow[d, "Fq"] \\
& FY
\end{tikzcd}$$

Proposition

The following functors admit precise factorizations w.r.t. $\mathcal{S}$:

1. Constant functors if $0 \in S$
2. Products of such functors if $\mathcal{S}$ closed under products
3. Coproducts of such functors if $\mathbb{C}$ extensive and $\mathcal{S}$ closed under coproducts
4. Right adjoints $R \iff$ the left adjoint L preserves $\mathcal{S}$-objects

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1. Polynomial functors
2. Analytic functors, e.g. the bag functor (finite multisets)
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Non-Example

Powerset $\mathcal{P}$ because $f(x) = \{y\} = \{y, y\}$

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- h coalgebra homomorphism $\Longrightarrow$ h open

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Definition

Path P of length n

$$I = P_0 \xrightarrow{p_0} FP_1 \quad P1 \xrightarrow{p_1} FP_2 \quad \cdots \quad FP_{n+1}$$

F-precise map F-precise map

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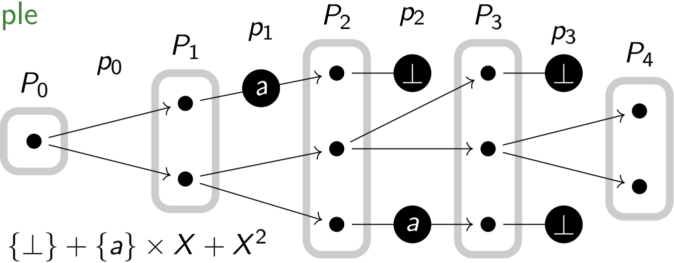
F-precise map

Example



$$FX = \{\perp\} + \{a\} \times X + X^2$$

Definition: Category Path(I, F)

Path P of length n

$$\begin{tikzcd}
I = P_0 \arrow[r, "p_0"] \arrow[d, "\phi_0"', "\cong"] & FP_1 \arrow[d, "F\phi_0"'] & P1 \arrow[r, "p_1"] \arrow[d, "\phi_1"', "\cong"] & FP_2 \arrow[d, "F\phi_1"'] & \cdots \arrow[d, phantom, "\cdots"] & FP_{n+1} \arrow[d, "F\phi_{n+1}"', "\cong"] \\
I = Q_0 \arrow[r, "q_0"] & FQ_1 & Q1 \arrow[r, "q_1"] & FQ_2 & \cdots & FQ_{n+1} & \cdots & FQ_{m+1}
\end{tikzcd}$$

prefix order

$m \geq n$

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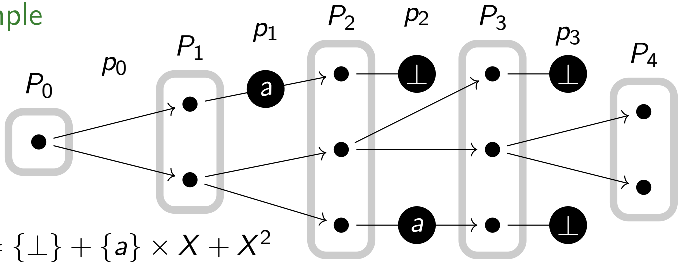

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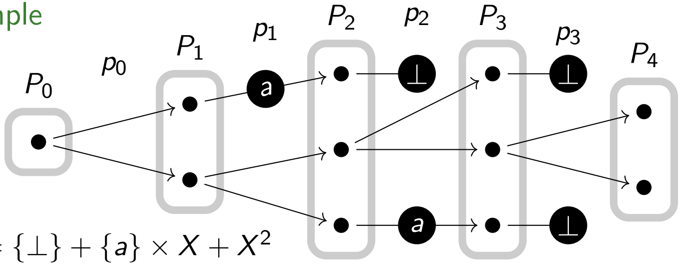

$FX = \{\perp\} + \{a\} \times X + X^2$

Relation to final chain of F

Truncation order

Full functor $\quad \mathrm{Path}(I, F) \twoheadrightarrow (\bigsqcup_{n\geq 0} \mathbb{C}(I, F^n 1), \leq)$

Main Result

Theorem

Branching

Input

Given:

- Functors $T, F \colon \mathbb{C} \to \mathbb{C}$ with order $\subseteq$ on $\mathbb{C}(X, TY)$
- F admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$ ✓
- $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\bot} 1$ plus axioms (T powerset-like) ?

Then there is a path category $\mathrm{Path}(I, F+1)$?
and for every $h \colon X \to Y$ in $\mathrm{LCoalg}(I, TF)$:

- h open & X reachable ? $\Longrightarrow$ h coalgebra homomorphism
- h coalgebra homomorphism $\Longrightarrow$ h open

Canonical Trace Semantics: $\mathrm{LCoalg}(I, TF) \to \coprod_{n \ge 0} \mathbb{C}(I, F^n 1)$

$\mathrm{trace}(X) = \{[w] \mid Jw \to X\}$ preserved by coalgebra hom.

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singletons

bottom

Pointings: $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\perp} 1$ plus Axioms (powerset-like)

Open map situation J: $\mathrm{Path}(I, F+1) \to \mathrm{LCoalg}(I, TF)$

$$\begin{tikzcd}
& P_0 \xrightarrow{p_0} FP_1 + 1 \quad P_1 \xrightarrow{p_1} FP_2 + 1 \arrow[d, mapsto, "J"'] & \\
P_0 + P_1 + P_2 \arrow[r] & F(P_1 + P_2) + 1 \arrow[r, "{[\eta,\perp]}"] & TF(P_0 + P_1 + P_2)
\end{tikzcd}$$

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Reachability

$$J\colon \mathrm{Path}(I, F+1) \longrightarrow \mathrm{LCoalg}(I, TF)$$

Definition: $(X, x_0, \xi) \in \mathrm{LCoalg}(I, TF)$ is reachable

..., if the runs $f\colon JP \to (X, x_0, \xi)$ are jointly surjective.

Theorem

(X, x_0, ξ) is reachable iff it has no proper subcoalgebra.

Coalgebraic definition
of reachability

Main Result

Theorem

Branching

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Instances	Tree Automata
$\mathbb{C}$	Set
$\mathcal{S} \subseteq \lvert\mathbb{C}\rvert$	all
I	1
T	$\mathcal{P}$, $\mathcal{P}_{\mathrm{f}}$
$F(X)$	analytic functors: polynomials Σ, finite multisets
$\mathbb{C}(I, F^n 1)$	Trees, height n
trace	Tree language

Instances	Tree Automata	Nominal Automata
$\mathbb{C}$	Set	Nominal Sets
$\mathcal{S} \subseteq \lvert\mathbb{C}\rvert$	all	strong ones
I	1	$\mathbb{A}^{\#k}$
T	$\mathcal{P}$, $\mathcal{P}_\mathrm{f}$	$\mathcal{P}_\mathsf{ufs}$, $\mathcal{P}_\mathrm{f}$
$F(X)$	analytic functors: polynomials Σ, finite multisets	$1 + [\mathbb{A}]X + \mathbb{A} \times X$ (RNNA) [Schröder, Milius Kozen, W, '17]
$\mathbb{C}(I, F^n 1)$	Trees, height n	bar-strings $\lvert$support$\rvert \leq k$
trace	Tree language	bar language

Instances	Tree Automata	Nominal Automata	Lasota's construction for $\mathbb{P} \hookrightarrow \mathbb{M}$ [Lasota '02]
$\mathbb{C}$	Set	Nominal Sets	Set$^{\|P\|}$
$\mathcal{S} \subseteq \|\mathbb{C}\|$	all	strong ones	all
I	1	$\mathbb{A}^{\#k}$	$I_0 = 1,\ I_P = \emptyset$
T	$\mathcal{P}$, $\mathcal{P}_{\mathrm{f}}$	$\mathcal{P}_{\mathsf{ufs}}$, $\mathcal{P}_{\mathrm{f}}$	$\mathcal{P}$ (per component)
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trace	Tree language	bar language	Composition has run

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What about **weighted systems**?

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Axioms on T restrict to $\mathcal{P}$..

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...still subsumes all open map situations

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Axioms on T restrict to $\mathcal{P}$..

Need: generalization of open maps

...still subsumes all open map situations

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References

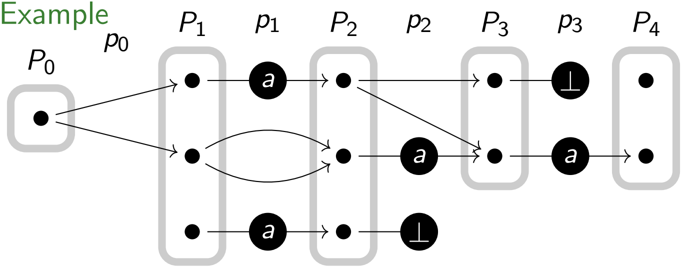
André Joyal, Mogens Nielsen, Glynn Winskel. “Bisimulation from Open Maps”. In: **Information and Computation** 127 (1996), pp. 164–185.

Sławomir Lasota. “Coalgebra morphisms subsume open maps”. In: **Theoretical Computer Science** 280.1 (2002). Coalgebraic Methods in Computer Science, pp. 123–135. ISSN: 0304-3975. DOI: `https://doi.org/10.1016/S0304-3975(01)00023-8`. URL: `http://www.sciencedirect.com/science/article/pii/S0304397501000238`.

Lutz Schröder, Dexter Kozen, Stefan Milius, Thorsten Wißmann. “Nominal Automata with Name binding”. In: **Proc. 20th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS 2017)**. Ed. by Javier Esparza, Andrzej Murawski. Vol. 10203. Lecture Notes Comput. Sci. (ARCoSS). Springer, 2017, pp. 124–142. URL: `http://arxiv.org/abs/1603.01455`.

$$FX = \{a\} \times X + X \times X,\ I = \{\bullet\},\ p_k\colon P_k \to FP_{k+1} + \{\perp\}$$

Non-Example



Tree automata in Sets

$I = 1$, T is $\mathcal{P}$ or $\mathcal{P}_{\mathrm{f}}$, F is analytic.

$$FX = \coprod_{\sigma/n \in \Sigma} X^n / G_\sigma$$

Path(I, F)

Path of length n = 'partial' F-tree of height n.

TF-coalgebra homomorphisms

... are open morphisms and thus preserve & reflect tree languages.

RNNA

Schröder, Kozen, Milius, Wißmann '17

TF-coalgebra for $T = \mathcal{P}_{\mathsf{ufs}}$ $FX = 1 + \mathbb{A} \times X + [\mathbb{A}]X$

$I := \mathbb{A}^{\#n}$, fixed $n \in \mathbb{N}$ $\mathcal{S} =$ Strong nominal sets

F-precise maps

- ... don't loose support
- ... don't loose order in the support
- if $f\colon \mathbb{A}^{\#n} \to FY$ is F-precise, then $Y = \mathbb{A}^{\#m}$ with $n \leq m \leq n + 1$.

Path(I, F)

Finite sequence of $F + 1$-precise maps

$\Rightarrow$ essentially bar strings

Lasota's construction for arbitrary $J\colon \mathbb{P} \hookrightarrow \mathbb{M}$ Lasota '02

Let $J0_{\mathbb{P}} = 0_{\mathbb{M}}$ and the pointing[a] $I = \chi^{0_{\mathbb{P}}}$ elsewhere.

$$\mathbb{F}\colon \mathsf{Set}^{|\mathbb{P}|} \to \mathsf{Set}^{|\mathbb{P}|} \qquad \mathbb{F}\colon (X_P)_{P\in\mathbb{P}} \mapsto \left(\prod_{Q\in\mathbb{P}} \mathcal{P}(X_Q)^{\mathbb{P}(P,Q)}\right)_{P\in\mathbb{P}}$$

Functor Beh: $\mathbb{M} \to \mathsf{LCoalg}(I, \mathbb{F})$, $M \mapsto (\mathbb{M}(P, M))_{P\in\mathbb{P}} \ldots$

---

[a] No pointing in [Las02]

$\mathbb{F} = T \cdot F$

$$T(X_P)_{P\in\mathbb{P}} = (\mathcal{P}X_P)_{P\in\mathbb{P}} \qquad F(X_P)_{P\in\mathbb{P}} = \Big(\coprod_{Q\in\mathbb{P}} \mathbb{P}(P,Q) \times X_Q\Big)_{P\in\mathbb{P}}$$

Path-category $\mathsf{Path}(I, F)$

$f\colon \chi^P \to FY$ F-precise iff $Y = \chi^Q$ for some $Q \in \mathbb{P}$

$\Rightarrow$ objects in $\mathsf{Path}(I, F)$ are: $0_{\mathbb{P}} \xrightarrow{m_1} P_1 \xrightarrow{m_2} P_2 \cdots \xrightarrow{m_n} P_n$

All the axioms

F $F+1$ admits precise factorizations, w.r.t. $\mathcal{S}$ and $I \in \mathcal{S}$

T If $(e_i\colon X_i \to Y)_{i\in I}$ jointly epic, then $f \cdot e_i \sqsubseteq g \cdot e_i$ for all $i \in I \Rightarrow f \sqsubseteq g$.

$[\eta, \bot]\colon \mathrm{Id}+1 \to T$, with $\bot_Y \cdot !_X \sqsubseteq f$ for all $f\colon X \to TY$

For every $f\colon X \to TY$, $X \in \mathcal{S}$,

$f = \bigsqcup\{[\eta,\bot]_Y \cdot f' \sqsubseteq f \mid f'\colon X \to Y+1\}$

$$\begin{tikzcd}
\forall A \in \mathcal{S} & A \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\sqsubseteq"] & TX \arrow[d, "Th"] & \overset{\exists x'}{\Longrightarrow} & A \arrow[rr, bend left, "x"] \arrow[r, dashed, "x'"] \arrow[dr, "y"'] & X+1 \arrow[r, "{[\eta,\bot]_X}"'] \arrow[d, "h+1"] \arrow[r, phantom, "\sqsubseteq", shift left=6, near start] & TX \arrow[d, "Th"] \\
& Y+1 \arrow[r, "{[\eta,\bot]_Y}"] & TY & & & Y+1 \arrow[r, "{[\eta,\bot]_Y}"] & TY
\end{tikzcd}$$
