

An Efficient Coalgebraic Paige Tarjan Algorithm

Thorsten Wißmann

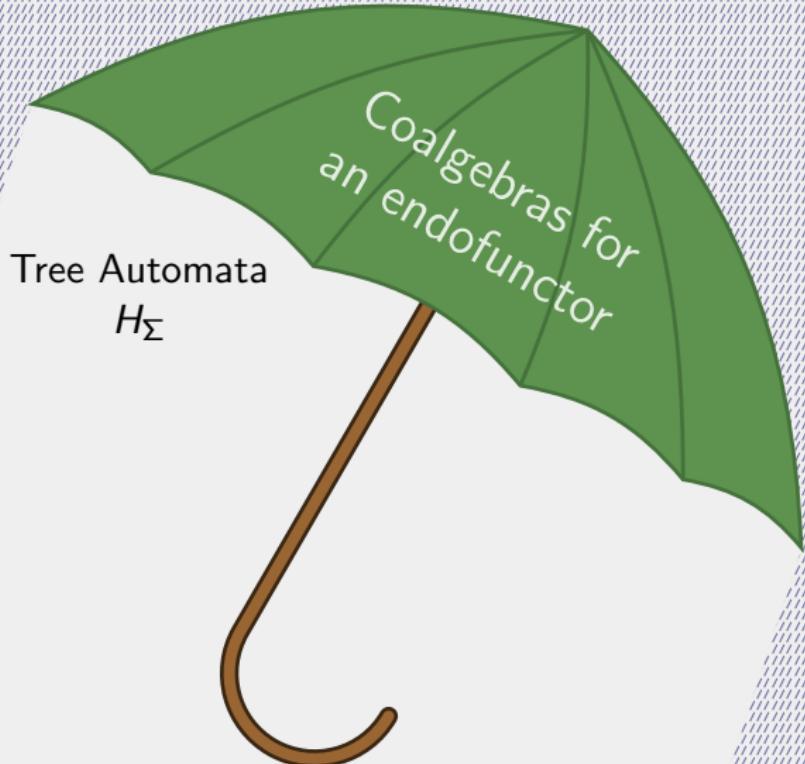
Joint work with:
Ulrich Dorsch, Stefan Milius, Lutz Schröder



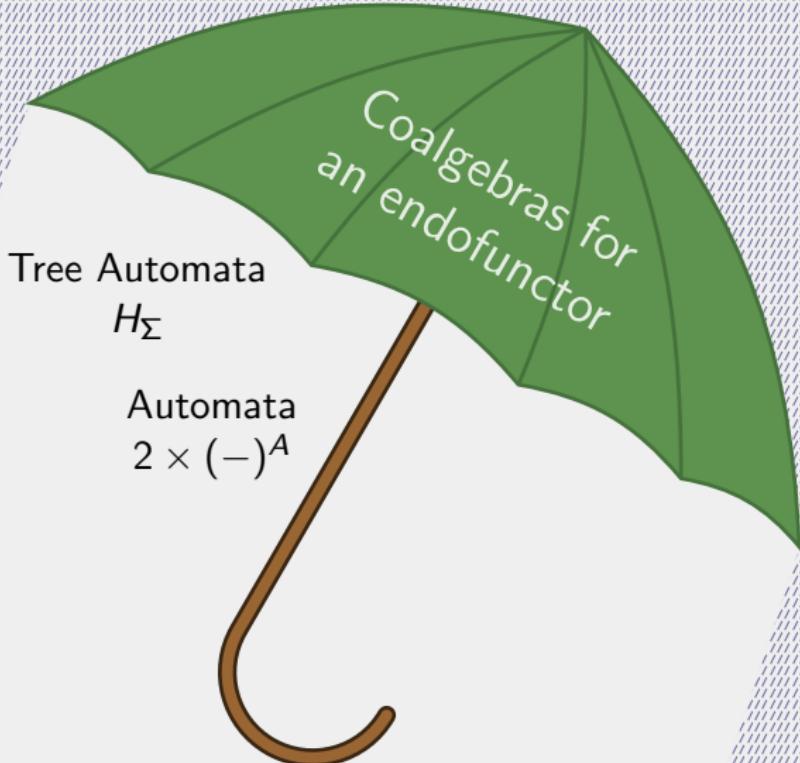
Calco Early Ideas
June 16, 2017

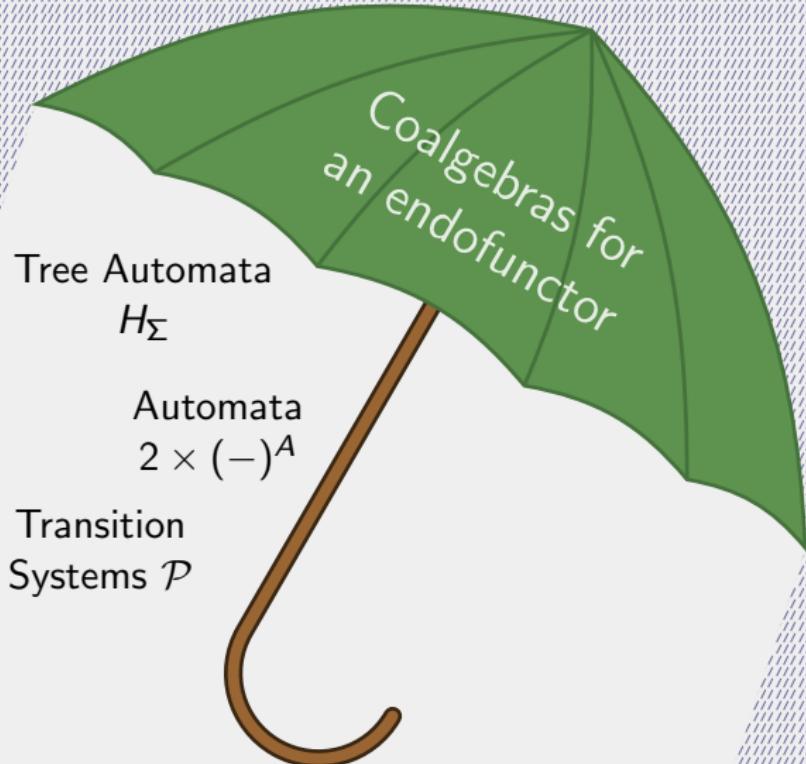


Efficient Minimization?

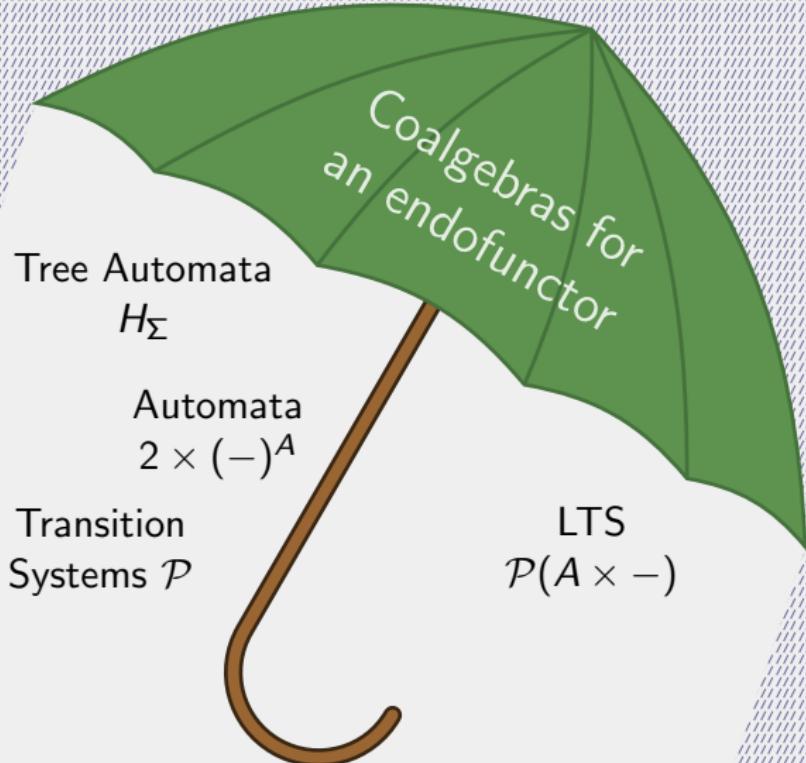


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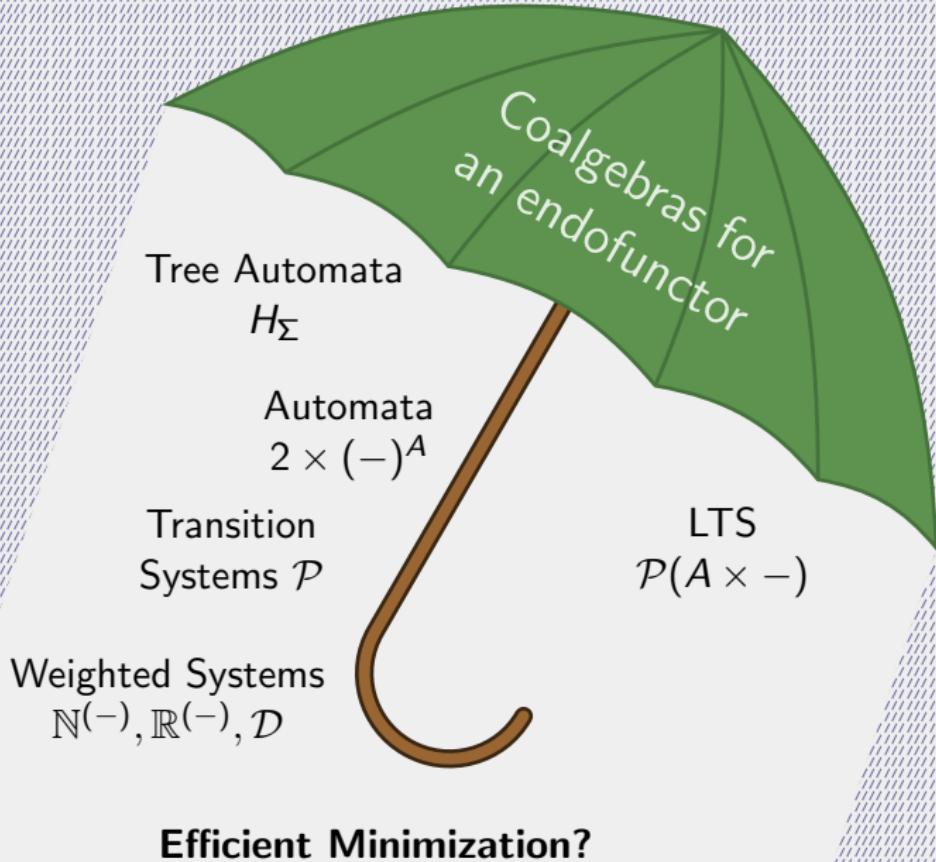


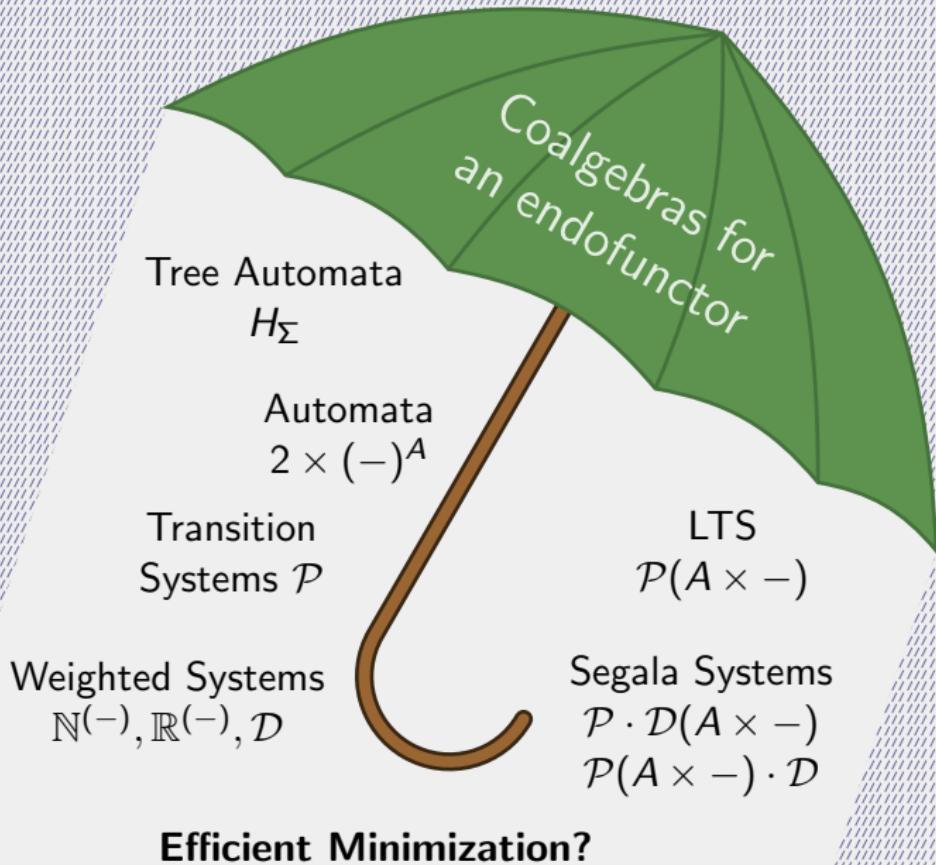


Efficient Minimization?



Efficient Minimization?





1. Assume
everything
equivalent

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 2. Have a quotient of X
- 

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 3. Unravel by one step
- 

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4. Pick some of the new information

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refine further

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- refine further

$$\begin{matrix} X \\ \downarrow ! \\ 1 \end{matrix}$$

1. Assume everything equivalent
 2. Have a quotient of X
 3. Unravel by one step
 4. Pick some of the new information
- refine further

$$\begin{array}{c} Q := \ker a \\ \Downarrow \\ X \\ \Downarrow \\ 1 \\ \Downarrow \\ \text{!} \\ \Downarrow \\ X \\ \Downarrow^a \\ A \end{array}$$

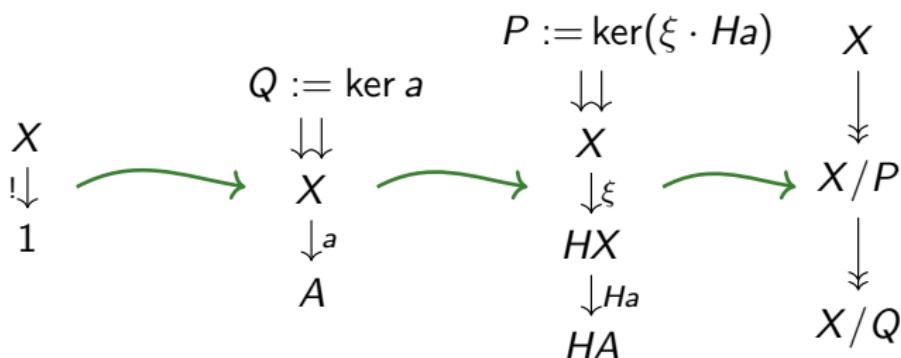
\rightarrow

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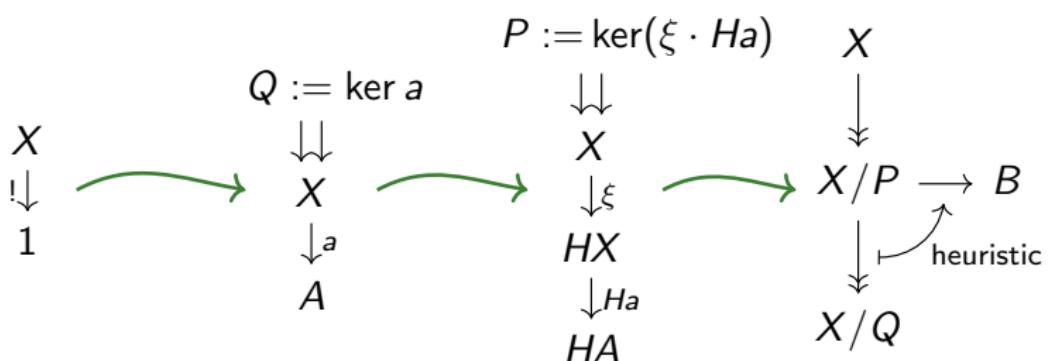
$$\begin{array}{c}
 P := \ker(\xi \cdot Ha) \\
 Q := \ker a \\
 X \downarrow \Downarrow \quad \quad \quad X \downarrow \xi \\
 1 \quad \quad \quad A \quad \quad \quad HX \downarrow Ha \\
 \quad \quad \quad \quad \quad \quad HA
 \end{array}$$

$\xrightarrow{\hspace{10em}}$

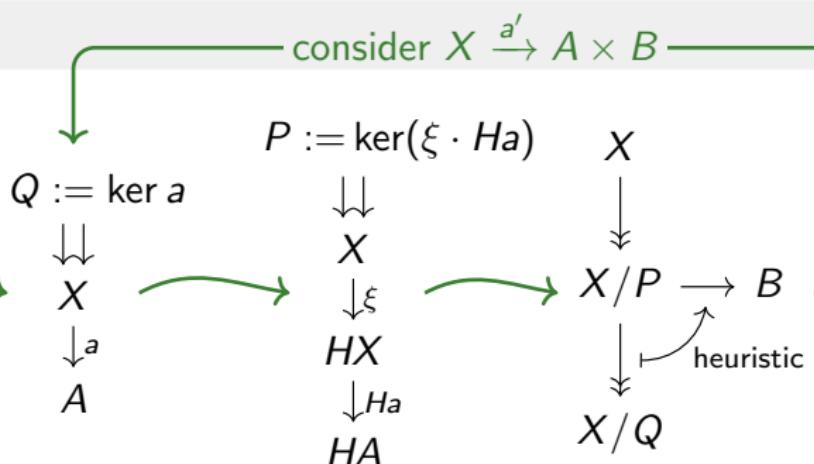
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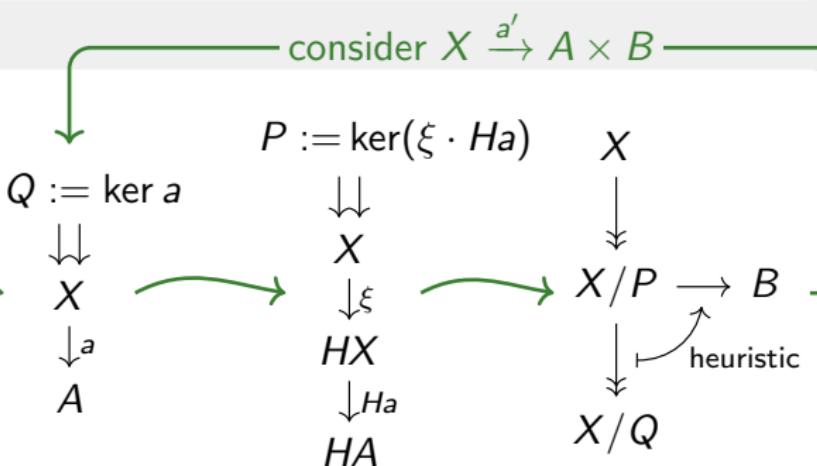
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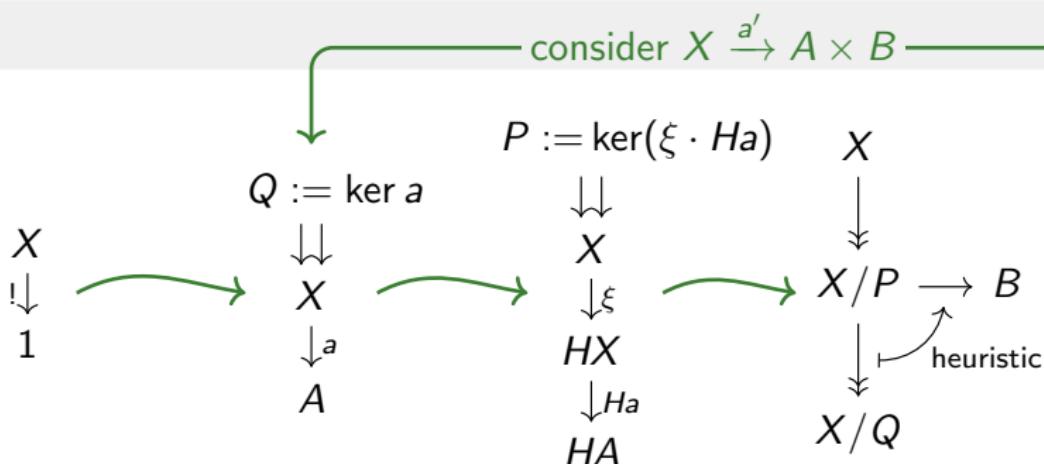


- refine further
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Heuristic id on X/P :
Use all immediately

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Heuristic id on X/P :
Use all immediately

Heuristic in Set:
Process “smaller half”

Assume

Finitely complete, H mono-preserving,
(RegularEpi,Mono)-factorisations

Theorem (Correctness)

$$\begin{array}{ccc} X & \xrightarrow{\xi} & HX \\ \downarrow & & \downarrow \\ X/P_i & \longrightarrow & H(X/Q_i) \end{array}$$

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If $P_i \cong Q_i$, then this

- ① is a coalgebra
- ② has no proper quotient

Incremental partitions

$Q := \ker a$

$$\begin{array}{c} \Downarrow \\ X \\ \downarrow^a \\ A \end{array}$$

Incremental partitions

$$\begin{array}{ccc} Q := \ker a & & Q \cap \ker b \\ \Downarrow & & \Downarrow \\ X & \xrightarrow{\quad} & X \\ \downarrow a & & \downarrow \langle a, b \rangle \\ A & & A \times B \end{array}$$

Incremental partitions

$$P := \ker(\xi \cdot Ha)$$

$$Q := \ker a$$

$$Q \cap \ker b$$

$$\begin{array}{c} \Downarrow \\ X \\ \downarrow a \\ A \end{array}$$

$$A \times B$$

$$\begin{array}{c} \Downarrow \\ X \\ \downarrow \xi \\ HX \\ \downarrow Ha \\ HA \end{array}$$



Incremental partitions

$Q := \ker a$	$Q \cap \ker b$	$P := \ker(\xi \cdot Ha)$???
\Downarrow X \Downarrow X \downarrow^a A	\Downarrow X \Downarrow X $\downarrow^{(a,b)}$ $A \times B$	\Downarrow X $\Downarrow \xi$ HX $\Downarrow Ha$ HA	\Downarrow X $\Downarrow \xi$ HX $\Downarrow H(a,b)$ $H(A \times B)$



Incremental partitions

$$\begin{array}{ccccccc}
 & & P := \ker(\xi \cdot Ha) & & \text{???} & & \\
 & & \Downarrow & & \Downarrow & & \\
 Q := \ker a & & Q \cap \ker b & & X & & X \\
 \Downarrow & & \Downarrow & & \Downarrow \xi & & \Downarrow \xi \\
 X & & X & & HX & & HX \\
 \downarrow a & & \downarrow \langle a, b \rangle & & \downarrow Ha & & \downarrow H\langle a, b \rangle \\
 A & & A \times B & & HA & & H(A \times B)
 \end{array}$$

Diagram illustrating the incremental construction of partitions. It shows three parallel rows of sets and their relationships through arrows and downarrows.

- Left Row:** $Q := \ker a$ (top), X (middle), A (bottom).
- Middle Row:** $Q \cap \ker b$ (top), X (middle), $A \times B$ (bottom).
- Right Row:** $P := \ker(\xi \cdot Ha)$ (top), X (middle), HX (middle), HA (bottom).
- Relationships:**
 - A green arrow points from Q to $Q \cap \ker b$.
 - A green arrow points from X to X (middle).
 - Downarrows connect Q to X , $Q \cap \ker b$ to X (middle), and X to A and $A \times B$.
 - Downarrows connect P to X , X to HX , HX to HA , and HA to $H(A \times B)$.
 - A red arrow points from HX to $H(A \times B)$.

Question: When is $\ker H\langle a, b \rangle = \ker\langle Ha, Hb \rangle$?

Incremental partitions

$Q := \ker a$	$Q \cap \ker b$	$P := \ker(\xi \cdot Ha)$???
\Downarrow X \Downarrow X \downarrow^a A	\Downarrow X \Downarrow X $\downarrow^{(a,b)}$ $A \times B$	\Downarrow X \downarrow^ξ HX \downarrow_{Ha} HA	\Downarrow X \downarrow^ξ HX $\downarrow^{H(a,b)}$ $H(A \times B)$

Theorem: In Set, $\ker H\langle a, b \rangle = \ker\langle Ha, Hb \rangle$ if

Incremental partitions

$$\begin{array}{ccccccc}
 & & & P := \ker(\xi \cdot Ha) & & \text{???} \\
 & & & \Downarrow & & \Downarrow \\
 Q := \ker a & & Q \cap \ker b & X & & X \\
 \Downarrow & & \Downarrow & \Downarrow \xi & & \Downarrow \xi \\
 X & \xrightarrow{\quad} & X & HX & & HX \\
 \downarrow a & & \downarrow \langle a, b \rangle & \downarrow Ha & & \downarrow H\langle a, b \rangle \\
 A & & A \times B & HA & & H(A \times B)
 \end{array}$$

Theorem: In Set, $\ker H\langle a, b \rangle = \ker\langle Ha, Hb \rangle$ if

$$\begin{array}{c}
 H(L + R) \\
 \downarrow \quad \text{monic} \quad \text{and} \\
 H(L+1) \times H(1+R)
 \end{array}$$

↑
“zippable”

Incremental partitions

$$\begin{array}{ccccccc}
 & & P := \ker(\xi \cdot Ha) & & \text{???} \\
 & & \Downarrow & & \Downarrow \\
 Q := \ker a & & Q \cap \ker b & X & X \\
 \Downarrow & & \Downarrow & \Downarrow \xi & \Downarrow \xi \\
 X & & X & HX & HX \\
 \downarrow a & & \downarrow \langle a, b \rangle & \downarrow Ha & \downarrow H\langle a, b \rangle \\
 A & & A \times B & HA & H(A \times B)
 \end{array}$$

Diagram illustrating the incremental construction of partitions. It shows three parallel rows of sets and their kernels. The first row starts with $Q := \ker a$, followed by $Q \cap \ker b$, then $P := \ker(\xi \cdot Ha)$, and ends with $H(A \times B)$. The second row starts with X , followed by X , then HX , and ends with HX . The third row starts with A , followed by $A \times B$, then HA , and ends with $H(A \times B)$. Green arrows indicate transitions between sets in adjacent columns: one from X to X , another from A to $A \times B$, and a curved arrow from P to HX .

Theorem: In Set, $\ker H\langle a, b \rangle = \ker\langle Ha, Hb \rangle$ if

$$\begin{array}{ccccc}
 H(L + R) & & & & \ker a \cup \ker b \\
 \downarrow & \text{monic} & \text{and} & & \text{a kernel} \\
 H(L+1) \times H(1+R) & & & & \\
 \uparrow & & & & \\
 \text{"zippable"} & & & &
 \end{array}$$

Diagram illustrating the "zippable" condition for the theorem. It shows two parallel rows. The top row starts with $H(L + R)$, followed by $\ker a \cup \ker b$. The bottom row starts with $H(L+1) \times H(1+R)$. A green arrow points from the bottom row up to the top row, labeled "zippable". Between the two rows, there is a column labeled "monic" and "and".

Incremental partitions

$$\begin{array}{ccccccc}
 & & P := \ker(\xi \cdot Ha) & & P \cap \ker(Hb \cdot \xi) & & \\
 & & \Downarrow & & \Downarrow & & \\
 Q := \ker a & & Q \cap \ker b & & X & & X \\
 \Downarrow & & \Downarrow & & \downarrow \xi & & \downarrow \xi \\
 X & & X & & HX & & HX \\
 \downarrow a & & \downarrow \langle a, b \rangle & & \downarrow Ha & & \downarrow H\langle a, b \rangle \\
 A & & A \times B & & HA & & H(A \times B)
 \end{array}$$

Diagram illustrating incremental partitions. It shows four sets of parallel arrows connecting sets \$A\$ and \$A \times B\$ to their respective kernels \$P\$ and \$P \cap \ker(Hb \cdot \xi)\$.

- Top row: \$Q := \ker a\$ and \$P := \ker(\xi \cdot Ha)\$. Arrows from \$A\$ to \$Q\$ and from \$A \times B\$ to \$P\$ are labeled \$\Downarrow\$.
- Middle row: \$Q \cap \ker b\$ and \$P \cap \ker(Hb \cdot \xi)\$. Arrows from \$A\$ to \$Q \cap \ker b\$ and from \$A \times B\$ to \$P \cap \ker(Hb \cdot \xi)\$ are labeled \$\Downarrow\$.
- Bottom row: \$X\$ and \$X\$. Arrows from \$A\$ to \$X\$ and from \$A \times B\$ to \$X\$ are labeled \$\downarrow \xi\$.
- Rightmost column: \$HX\$, \$H(X \cap \ker b)\$, and \$H(A \times B)\$. Arrows from \$X\$ to \$HX\$ and from \$X\$ to \$H(X \cap \ker b)\$ are labeled \$\downarrow \xi\$.

Green arrows indicate specific relationships between the sets:

- A green arrow points from \$Q\$ to \$Q \cap \ker b\$.
- A green arrow points from \$A\$ to \$A \times B\$.
- A green arrow points from \$P\$ to \$P \cap \ker(Hb \cdot \xi)\$.

Theorem: In Set, $\ker H\langle a, b \rangle = \ker\langle Ha, Hb \rangle$ if

$$\begin{array}{ccccc}
 H(L + R) & & & & \ker a \cup \ker b \\
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 H(L+1) \times H(1+R) & & & & \\
 \uparrow & & & & \\
 \text{"zippable"} & & & &
 \end{array}$$

Diagram illustrating the conditions for the theorem. It shows a commutative diagram where \$H(L+R)\$ is split into \$H(L+1) \times H(1+R)\$ via a monic map \$\downarrow\$. An upward arrow labeled "zippable" connects \$H(L+1) \times H(1+R)\$ back to \$H(L+R)\$. To the right, the condition \$\ker a \cup \ker b\$ is given as a kernel.

Setting for complexity analysis

Category:

Set

Heuristic:

smaller half

Functor:

zippable &
encoding

Setting for complexity analysis

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Assumption: Functor encoding

- coalgebra structure as edges with labels

$$X \xrightarrow{\xi} HX \xrightarrow{\flat} \mathcal{P}(L \times X)$$

- compute “smaller half” heuristic in linear time

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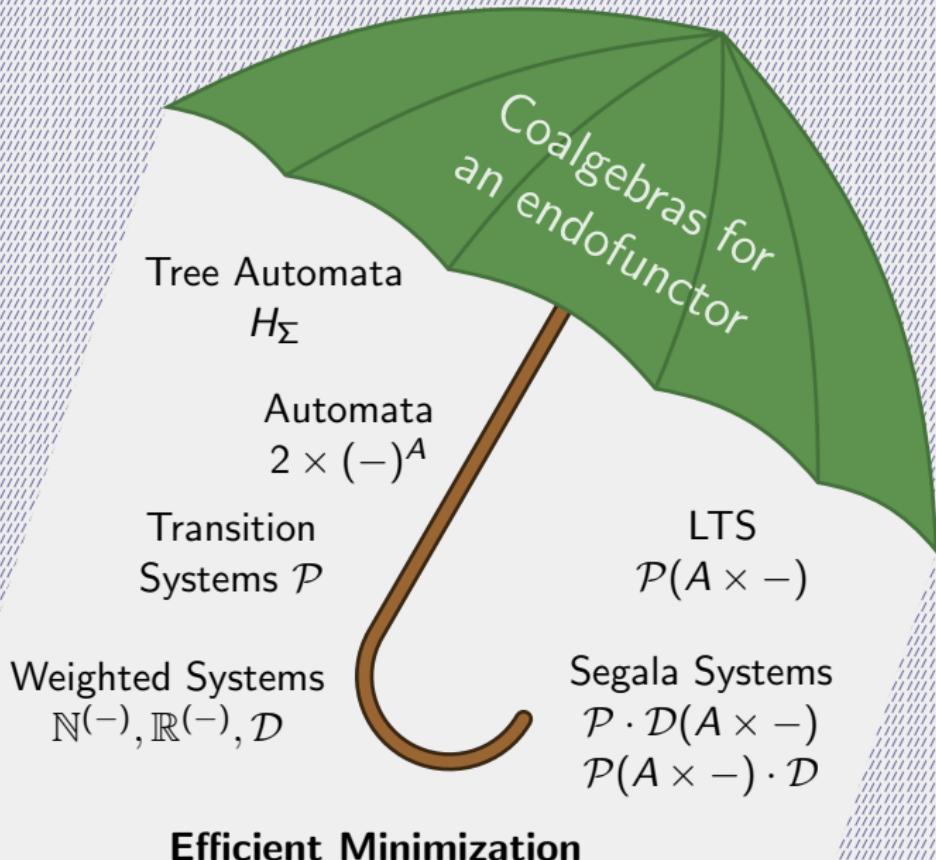
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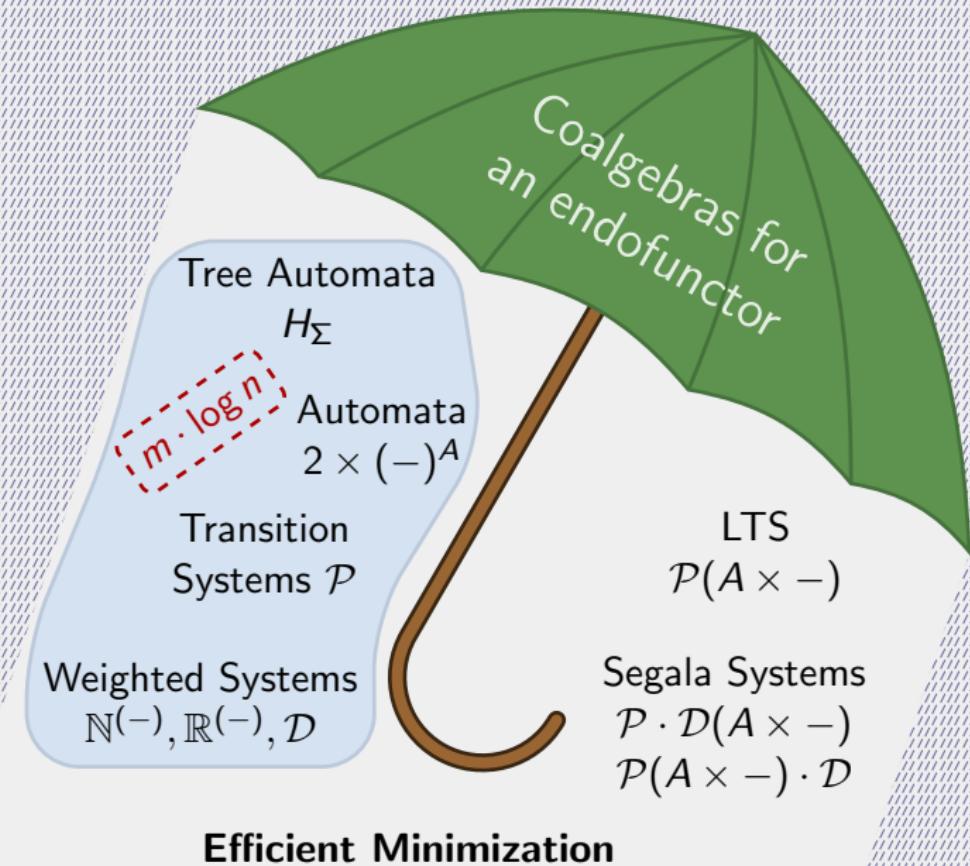
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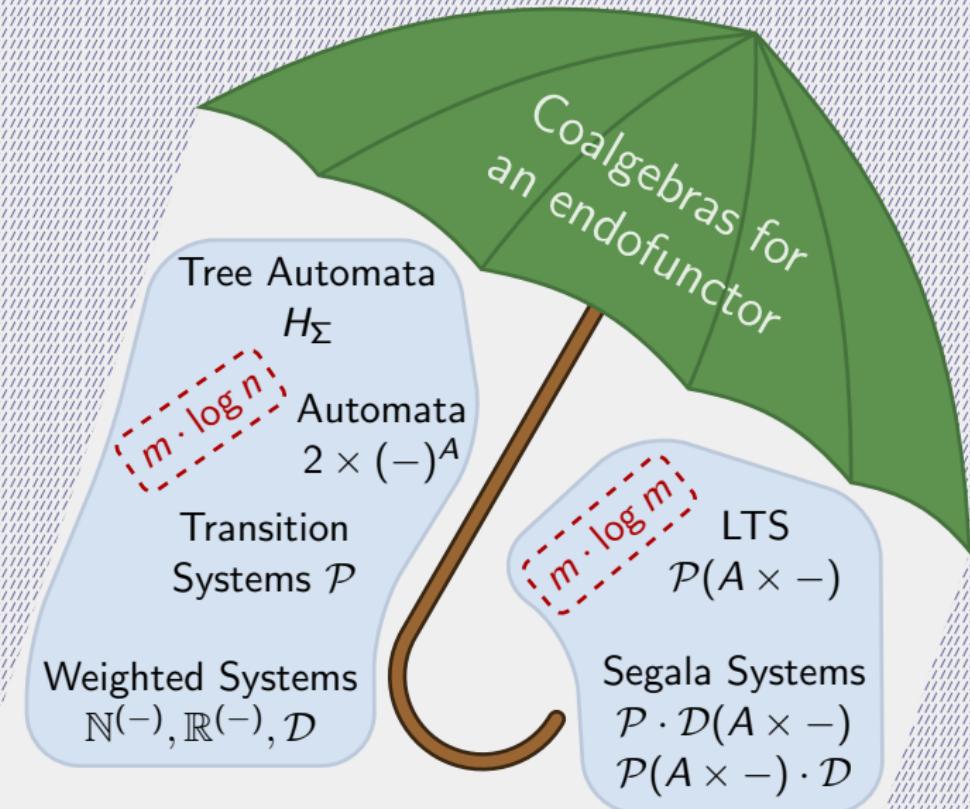
- compute “smaller half” heuristic in linear time

Theorem

Overall complexity: $\mathcal{O}((m+n) \cdot \log n)$ for $n = |X|$, $m = \sum_{x \in X} |\flat\xi(x)|$







Efficient Minimization

Functors H zippable, if

$H(L + R) \xrightarrow{\text{unzip}} H(L + 1) \times H(1 + R)$ is monic.

E.g. Id, Constants, \times , $+$, \hookrightarrow , $M^{(-)}$, part. additive

Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{-\}$

$$\begin{array}{cccccc} a_1 & a_2 & b_1 & a_3 & b_2 \\ & & \swarrow & & \uparrow \text{unzip} \\ (a_1 a_2 - a_3 -, & & & & & \end{array}$$

$\dots - b_1 - b_2)$

$(-)^*$ is zippable

$$\begin{array}{c} \{a_1, a_2, b_1\} \\ \swarrow \uparrow \text{unzip} \\ (\{a_1, a_2, -\}, \\ \{-, b_1\}) \end{array}$$

\mathcal{P} is zippable

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$H(L + R) \xrightarrow{\text{unzip}} H(L + 1) \times H(1 + R)$ is monic.

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Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{-\}$

$a_1 \ a_2 \ b_1 \ a_3 \ b_2$ $\xrightarrow{\text{unzip}}$

$(a_1 a_2 - a_3 -, - b_1 - b_2)$

$(-)^*$ is zippable

$\{a_1, a_2, b_1\} \xrightarrow{\text{unzip}}$

$(\{a_1, a_2, -\}, \{-, b_1\})$

\mathcal{P} is zippable

$\{\{a_1, b_1\}, \{a_2, b_2\}\}$

$\{\{a_1, b_2\}, \{a_2, b_1\}\}$

$\text{unzip} \xrightarrow{\quad} (\{\{a_1, -\}, \{a_2, -\}\}, \{-, b_1\}, \{-, b_2\})$

\mathcal{PP} is not zippable

~~Composition~~

~~Quotients~~

$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

$\Leftrightarrow \ker a \cup \ker b$ transitive

$\Leftrightarrow \forall x \in X : [x]_a \subseteq [x]_b \text{ or } [x]_a \supseteq [x]_b$

Example



Non-Example



$$A \xleftarrow{a} X \xrightarrow{b} B$$

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Example



$X / \ker a$



$X / \ker b$

Non-Example



$X / \ker a$



$X / \ker b$

Process smaller half for $X \xrightarrow{f} F \xrightarrow{g} G$

Find $x \in X$, with $S := [x]_f$, $C := [x]_{gf}$, such that $2 \cdot |S| \leq |C|$.

Return $\langle \chi_S, \chi_C \rangle : X \rightarrow 2 \times 2$

Functor encoding

- internal weights $W, w : HX \rightarrow \mathcal{P}X \rightarrow W$
- edge labels L
- $\flat : HX \rightarrow \mathcal{B}_f(L \times X)$
- update : $\mathcal{B}_f(L) \times W \longrightarrow W \times H(2 \times 2) \times W$



Functor:	$G^{(-)}$	\mathcal{B}_f	\mathcal{D}	\mathcal{P}	H_Σ
Labels L :	G	\mathbb{N}	$[0, 1]$	1	\mathbb{N}
Weights W :	$G^{(2)}$	$\mathcal{B}_f 2$	$\mathcal{D} 2$	\mathbb{N}	$H_\Sigma 2$
$w(C), C \subseteq Y$:	$G \chi_C$	$\mathcal{B}_f \chi_C$	$\mathcal{D} \chi_C$	$ C \cap (-) $	$H_\Sigma \chi_C$

Future work

- Implementation & Benchmarking
- $\mathcal{O}(m \cdot \log n)$ on $\mathcal{P}(A \times -)$
- $\ker\langle Ha, Hb \rangle = \ker H\langle a, b \rangle$ outside of Set?
- Further functors, e.g. monotone neighbourhoods.