Semantics of Nondeterministic Automata in a Topos

As presented by Philip Kaluđerčić

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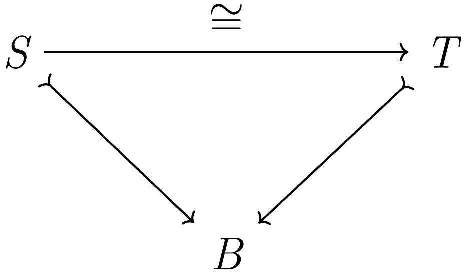
Part I

How to express a Nondeterministic Automaton in Categorical Terms?

Some preliminaries

Definition

A *subobject* is a isomorphism class of monos.



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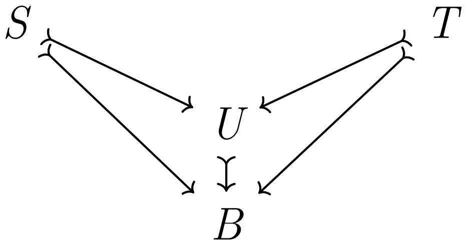
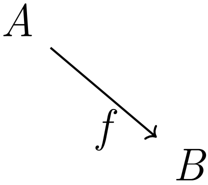


where $\mathrm{Sub}_{\mathscr{E}}(B)$ is the partial order of B subobjects.

Some more preliminaries

Definition

A *(epi,mono)-factorisation* of f: $A \to B$ is



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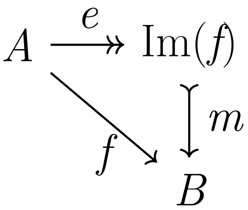

where m is the subobject of the image of f.

A $\mathscr{C}$-Automaton consists of[1]...

$$Q$$

A object of states

[1] Frank, Milius, and Urbat 2023.

A $\mathscr{C}$-Automaton consists of[1]...

Q Σ

A object of the input alphabet

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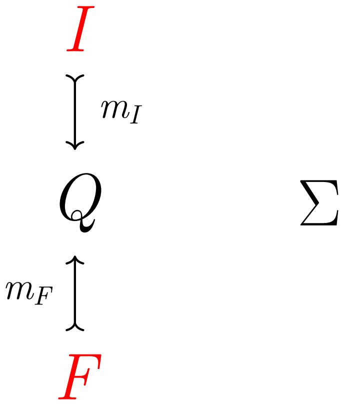
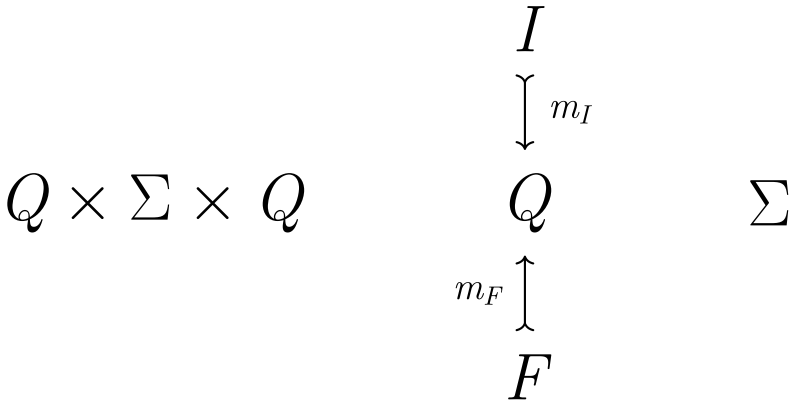


A subobject of initial and final states

[1] Frank, Milius, and Urbat 2023.

A $\mathscr{C}$-Automaton consists of[1]...



Of all possible transitions...

[1] Frank, Milius, and Urbat 2023.

A $\mathscr{C}$-Automaton consists of[1]…

$$
\begin{array}{ccc}
\color{red}{\delta} & I & \\
\downarrow m_\delta & \downarrow m_I & \\
Q \times \Sigma \times Q & Q & \Sigma \\
 & \uparrow m_F & \\
 & F &
\end{array}
$$

…a subobject of legal transitions

[1] Frank, Milius, and Urbat 2023.

Construction of accepted runs

$$\begin{array}{c}\delta^n\\ \big\downarrow {\scriptstyle m_\delta^n}\\ (Q\times\Sigma\times Q)^n\end{array}$$

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$$\begin{array}{ccc} & & \delta^n \\ & & \big\downarrow{\scriptstyle m_\delta^n} \\ I \times (\Sigma \times Q)^{n-1} \times \Sigma \times F & \xrightarrow{d_{n,A}} & (Q \times \Sigma \times Q)^n \end{array}$$

Construction of accepted runs

$$\begin{array}{ccc} \mathsf{AccRun}_A^{(n)} & \overset{\bar{d}_{n,A}}{\rightarrowtail} & \delta^n \\ \big\downarrow{\scriptstyle \overline{m}_\delta^n} & \lrcorner & \big\downarrow{\scriptstyle m_\delta^n} \\ I \times (\Sigma \times Q)^{n-1} \times \Sigma \times F & \overset{d_{n,A}}{\rightarrowtail} & (Q \times \Sigma \times Q)^n \end{array}$$

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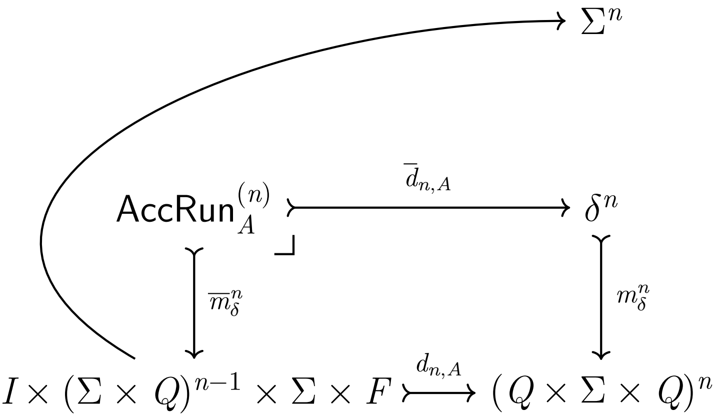

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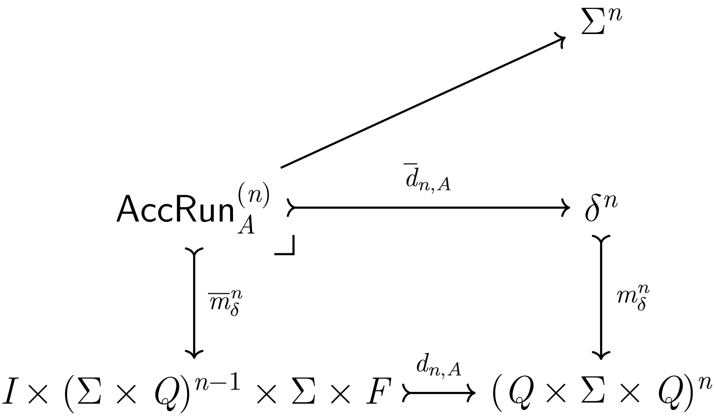

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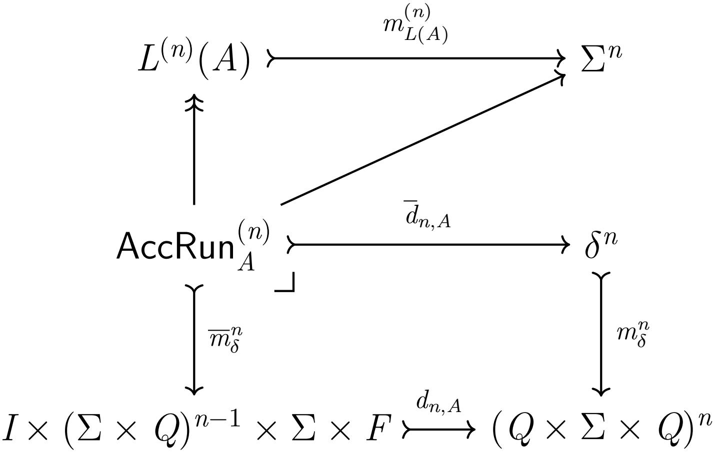
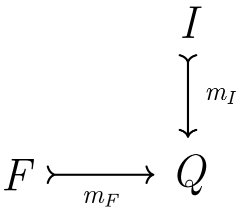


...and for the empty word



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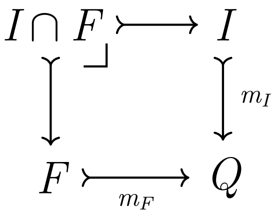

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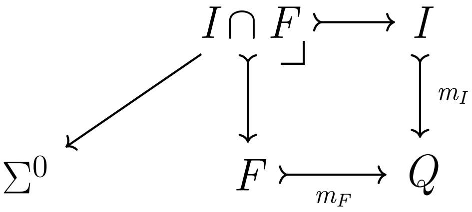

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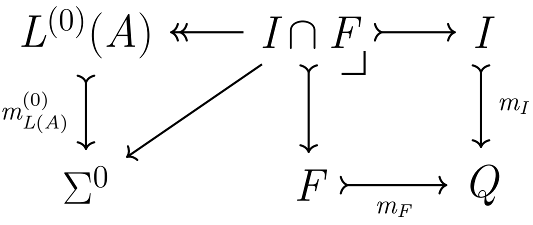

Part II

What is a Topos?

A category $\mathscr{E}$ is a topos *topos* iff

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- It is finitely complete,
- It is a CCC (has exponentials),
- It has a subobject classifier.

$$\begin{array}{c} 1 \\ \big\downarrow \mathrm{true} \\ \Omega \end{array}$$

$$\begin{array}{ccc} S & \qquad & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{true}} \\ B & & \Omega \end{array}$$

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{true}} \\ B & \xrightarrow[\;\phi\;]{} & \Omega \end{array}$$

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$$\text{Hom}_{\mathscr{E}}(B, \Omega) \cong \text{Sub}_{\mathscr{E}}(B)$$

Definition

A *power object* $\mathbf{P}\,A$ of an object A,

$$B \times A \xrightarrow{\;f\;} \Omega$$

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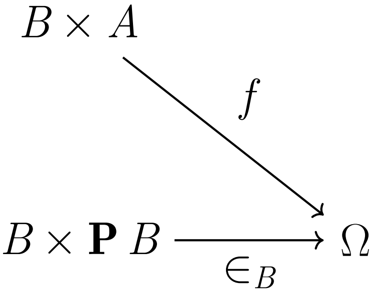

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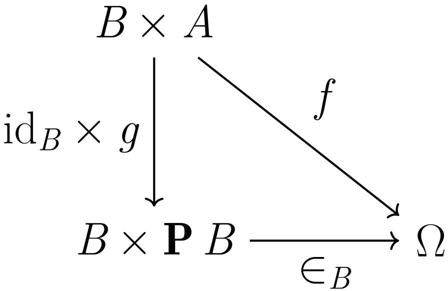

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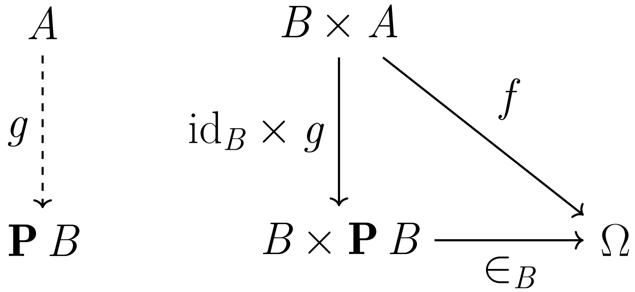

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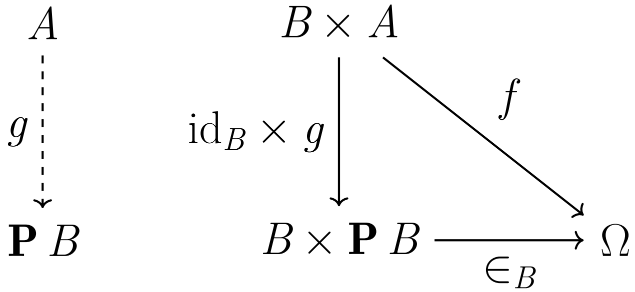

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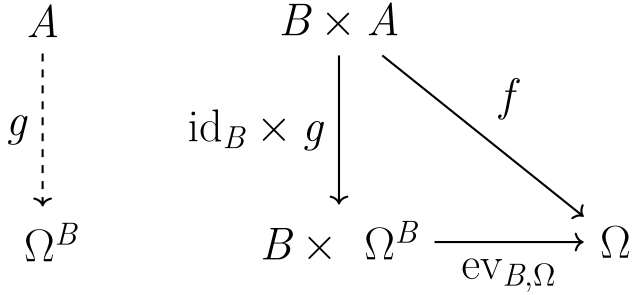

$$\mathrm{Hom}_{\mathscr{E}}\,(A \times B, \Omega) \cong \mathrm{Hom}_{\mathscr{E}}(A,\ \Omega^B)$$

Power-objects are the generalisation of power-sets in a topos.

What a Topos gives us for "free"?

- Finitely Cocomplete,
- All (epi,mono)-factorisations,
- A power-object functor
- An internal language.

Definition

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In a CCC this language is λ-Calculus with products *or* intuitionistic minimal logic.
Objects corresponds to {types,propositions} and morphisms to {terms,proofs}.

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In a topos this language is higher-order, intuitionistic, finitary set theory.

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Example (Char. morphism of S)

Given $m\colon S \rightarrowtail B$, we can define a predicate

$$\phi_m \coloneqq \lambda\, b.\,(b \in S)\colon\ B \to \Omega$$

as the characteristic morphism.

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Example (Extension of ϕ)

Given $\phi\colon B \to \Omega$, we can define

$$S \coloneqq \{\, b\colon B \mid \phi(b) = \mathrm{true} \,\} : \mathbf{P}\, B$$

as the subobject $m\colon S \rightarrowtail B$.

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Example (An internal image)

Given $f\colon A \to B$, we can define

$$\mathrm{Im}(f) \coloneqq \{\, b\colon B \mid \exists\, a \in A.\, f(a) = b \,\}$$

as the (epi,mono)-factorisation.

Definition

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Example (Intersection of subobjects)

Given $m\colon S \rightarrowtail B$ and $n\colon T \rightarrowtail B$, w.c.d.

$$S \cap T \coloneqq \{\, b\colon B \mid \phi_m(b) \wedge \phi_n(b) \,\}$$

as the pullback of m and n.

Part III

How to describe the Semantics of a $\mathscr{C}$-Automaton internally?

First, the accepted runs

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$$\mathsf{AccRun}_A^{(n)} \coloneqq \mathrm{Im}(d_{n,A}) \cap \mathrm{Im}(m_\delta^n)$$

where $n \geq 1$

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where $n \geq 1$, and

$${\pi_2}^n(a) \coloneqq \pi_2(\pi_1(a))\pi_2(\pi_2(a)) \ldots \pi_2(\pi_{n-1}(a))\pi_2(\pi_n(a))$$

Part IV

How to describe the Semantics of a Coalgebra in a Topos?

The Coalgebra in Question

$$FQ = \Omega \times (\mathbf{P}\ Q)^{\Sigma}$$

[2] Jacobs, Silva, and Sokolova 2012.

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By reusing a result involving Eilenberg-Moore Algebras by Jacobs, Silva, and Sokolova[2], we can show that the final coalgebra of F is $(\mathbf{P}\,\Sigma^*)$.

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So assume that $\mathscr{E}$ has countable coproducts...

Given a coalgebra $\langle o, t \rangle : Q \longrightarrow \Omega \times (\mathbf{P}\ Q)^{\Sigma}$, Given a coalgebra $\langle o, t \rangle : Q \rightarrow \Omega \times (\mathbf{P}\ Q)^{\Sigma}$, we can now give a unique homomorphism

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Part V

Do the descriptions match?

Coalgebra $\leftrightsquigarrow$ $\mathscr{C}$-Automaton?

Given a $A = (Q, \Sigma, I, F, \delta)$, we can define a coalgebra $\langle o, t \rangle : Q \longrightarrow \Omega \times (\mathbf{P}\ Q)^{\Sigma}$:

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To handle multiple initial states, we adjust

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$$[\![I]\!]^* \coloneqq \left\{ w \in \Sigma^* \;\middle|\; \exists\, q \in I.\ o(\overline{t(q)}(w)) \right\}$$

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$$[\![I]\!]^*_{n} \coloneqq \left\{ w \in \Sigma^{n} \;\middle|\; \exists\, q \in I.\ o(\overline{t(q)}(w)) \right\}$$

Given a $\mathscr{C}$-Automaton A...

For any accepting run a: $\mathsf{AccRun}_A^{(n)}$,

$$(q_1, s_1, q_2), (q_2, s_2, q_3), \ldots, (q_n, s_n, q_{n+1}),$$

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Given a Coalgebra $\langle o, t \rangle$...

For some $q_1 \in I$ and chain of transitions

$$q_1 \xrightarrow{t(s_1)} q_2 \xrightarrow{t(s_2)} \ldots \xrightarrow{t(s_n)} q_{n+1}$$

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$$(q_n, s_n, q_{n+1}) = \pi_3(\mathsf{AccRun}_A^{(n)})$$

Part VI

What if we don't want to be countably extensive?

Definition

A *graded monad* is a family of endofunctors

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The laborious part!

Example (Depth-Limited Semantics)

$$M \coloneqq \left(\mathbf{P}\left(\Sigma^{<n} + - \times \Sigma^{n}\right)\right)_{n\in\mathbb{N}}$$

[3] Milius, Pattinson, and Schröder 2015.

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$$M \coloneqq \left(\mathbf{P}\left(\Sigma^{<n} + - \times \Sigma^{n}\right)\right)_{n\in\mathbb{N}}$$

gives the α-*pretrace sequence*[3] of coalgebra γ

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using which we can define a semantic map

$$[\![I]\!]_n^\gamma \coloneqq \left\{\ w\colon \Sigma^n\ \middle|\ \exists\, q \in I.\, \iota_1(w) \in \gamma^{(n+1)}(q)\ \right\}.$$

[3]Milius, Pattinson, and Schröder 2015.

$$\gamma^{(0)}(q) =\ \{\ \iota_2(\epsilon,\, q)\ \}$$

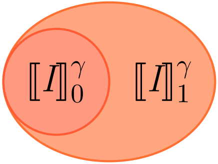
$$\gamma^{(n)} \colon\ Q \longrightarrow \mathbf{P}\left(\Sigma^{<n} + \Sigma^{n} \times\ Q\right)$$

$$\llbracket I \rrbracket_0^{\gamma}$$

$$\gamma^{(1)}(q) = \ \{\, \iota_1(\epsilon) \mid o(q) \,\}$$

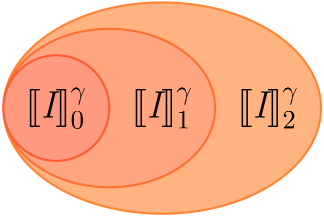
$$\cup \ \left\{ \iota_2(sw, p) \,\middle|\, \exists\, q' \in t(q)(s).\ \iota_2(w, p) \in \gamma^{(0)}(q') \right\}$$

$\gamma^{(n)}\colon\ Q \longrightarrow \mathbf{P}\left(\Sigma^{<n} + \Sigma^{n} \times\ Q\right)$



$$\begin{aligned}\gamma^{(2)}(q) = &\ \{\,\iota_1(\epsilon) \mid o(q)\,\}\\ &\cup\ \left\{\,\iota_1(sw)\ \middle|\ \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(1)}(q')\,\right\}\\ &\cup\ \left\{\,\iota_2(sw,p)\ \middle|\ \exists\, q' \in t(q)(s).\, \iota_2(w,p) \in \gamma^{(1)}(q')\,\right\}\end{aligned}$$

$$\gamma^{(n)} \colon Q \longrightarrow \mathbf{P}\left(\Sigma^{<n} + \Sigma^{n} \times Q\right)$$



$$\begin{aligned}\gamma^{(3)}(q) = \; & \{\, \iota_1(\epsilon) \mid o(q) \,\} \\ \cup \; & \left\{\, \iota_1(sw) \,\middle|\, \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(2)}(q') \,\right\} \\ \cup \; & \left\{\, \iota_2(sw, p) \,\middle|\, \exists\, q' \in t(q)(s).\, \iota_2(w, p) \in \gamma^{(2)}(q') \,\right\}\end{aligned}$$

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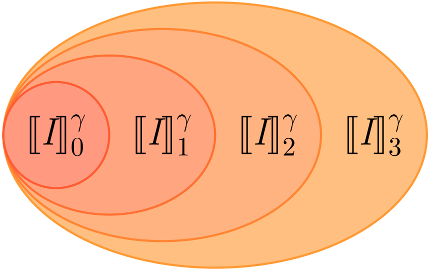

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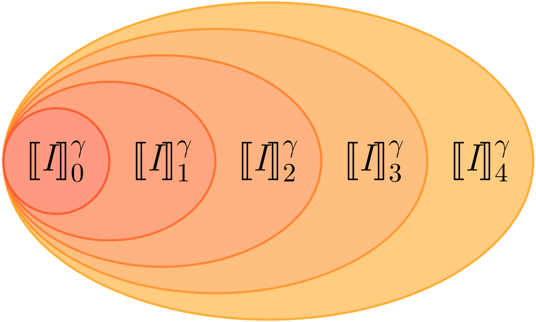

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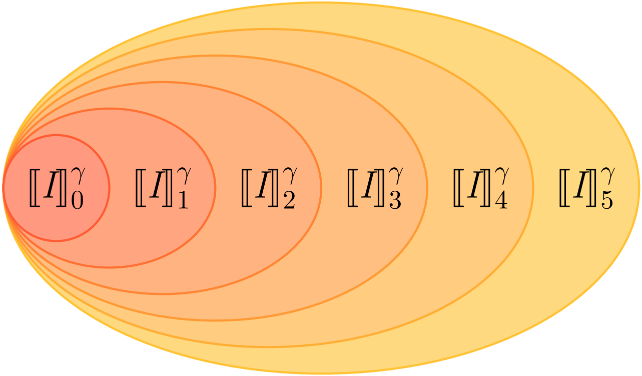

$$\begin{aligned}\gamma^{(6)}(q) = \; & \{\, \iota_1(\epsilon) \mid o(q) \,\} \\ \cup \; & \left\{ \iota_1(sw) \;\middle|\; \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(5)}(q') \right\} \\ \cup \; & \left\{ \iota_2(sw, p) \;\middle|\; \exists\, q' \in t(q)(s).\, \iota_2(w, p) \in \gamma^{(5)}(q') \right\}\end{aligned}$$

Part VII

Do these also match?

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From this data we can construct a acc. run

$$(q_1, s_1, q_2), (q_2, s_2, q_3), \ldots, (q_m, s_m, q_{m+1})$$

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and hence

$$s_1 s_2 \ldots s_m \in [\![I]\!]_n^\gamma.$$

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