Friedrich-Alexander-Universität Erlangen-Nürnberg

Semantics of Categorical Nondeterministic Automata in a Topos

Master's Thesis at the Chair for Theoretical Computer Science

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Erlangen, 2025/01/27, last typeset 26th January 2025 using LaTeX 2_ε

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Erlangen, 26th January 2025

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1 Introduction

Category theory is a fruitful ground to discuss the structures of mathematical objects and expose their common properties. To take an example that goes back to Grothendieck [Gro61], it is possible to define a group *in* a category, by selecting a group object $G \in \mathsf{Ob}(\mathscr{C})$ along with the morphisms

$$m: G \times G \longrightarrow G \qquad e: 1 \rightarrowtail G \qquad (-)^{-1}: G \longrightarrow G,$$

that satisfy associativity, unitality and invertability, all of which can be stated as commutative diagrams [Awo06, Def. 4.1, p. 75]. The advantage of this approach, as opposed to the standard "Bourbaki approach", is that the abstraction yields different results depending on the category: In **Sets** we arrive back at the conventional definition of groups, in **Grp** we get Abelian groups, while in the category of smooth manifolds we arrive at Lie groups. Results proven for a group in an (arbitrary) category can are applicable to each concrete instance.

Another example is the theory of functor coalgebras. These provide a general framework for discussing state based systems, capable of deriving general results whilst being parametric over a functor:

$$\underbrace{FX = A \times X}_{\text{Streams of } A\text{s}} \qquad \underbrace{FX = A \times X^B}_{\text{Moore automata}} \qquad \underbrace{FX = (A \times X)^B}_{\text{Mealy automata}} \qquad \underbrace{FX = \mathbf{2} \times \wp(X)^\Sigma}_{\text{Non-deterministic automata}}$$

The topic at hand will be nondeterministic automata in a category (much alike the example of internalising groups given above) and their semantics. The notion was introduced by Frank, Milius and Urbat [FMU23, Definition 6.5 & 6.6, p. 48:11], and generalises nondeterministic automata to arbitrary categories with sufficient structure. In **Sets** this results in conventional nondeterministic automata while in **Nom** it describes nondeterministic orbit-finite nominal automata.

We will be working within (elementary) toposes, kinds of categories that exhibit an "internal language" with set-like properties. This allows us to simplify our proofs by interpreting categorical statements in the language of set theory.

We can now state the goal of this thesis: The aforementioned paper conjectures that inside a topos, the accepted language of categorical automata coincides with the trace semantics of F-coalgebras. We will begin by introducing the topos-theoretical preliminaries as well as the precise definition of a categorical automaton in Chapter 2. Chapter 3 equates internal definitions of both the trace semantics of a coalgebra and a categorical automaton. This requires the additional assumption that the category has countable coproducts, which is not always the case, as toposes only ensure finite colimits. A second approach in Chapter 4 will involve graded monads and a depth-limited semantics, which is compatible with the finitist universe of a topos.

2 Technical Prolegomena

We will begin by defining "toposes" and their "internal logic". After that, we will show how to internalise a nondeterministic automaton inside a category.

We assume the reader is familiar with category theory, including concepts such as limits, colimits, functors, natural transformations, and exponentials. If this not the case, we recommend consulting the relevant literature. Suggestions include introductions by Mac Lane [Mac13], Awodey [Awo06] and Pierce [Pie91].

2.1 A Brief Introduction to Topos Theory

An informal definition of a *(elementary) topos* is a category with sufficient structure to serve as a model for intuitionistic set theory. This fact allows us to soundly reason about categorical statements in the *language* of set theory. Before proceeding to toposes, will recall another standard example of a category with some structure:

Definition 2.1. A Cartesian closed category (CCC) is a category $\mathscr{C}$ with

1. a terminal object 1,
2. all binary products $A \times B$ for $A, B \in \mathsf{Ob}(\mathscr{C})$,
3. all exponentials B^A for $A, B \in \mathsf{Ob}(\mathscr{C})$.

Example 2.2. Categories that exhibit the sufficient properties to be CCCs include the category of sets and functions **Sets**, of finite sets **FinSets**, the category of G-sets of a group G, the category of presheafs $\mathbf{Sets}^{\mathscr{C}}$ and the category of Heyting algebras. A counterexample is the general category of topological spaces **Top**, as this does not have *all* exponentials.

This section *formally* introduces the necessary definitions for the subsequent chapters. At the same time, it should also serve as a general introduction to topos theory, for any reader interested in the "set-like" and logical aspects of the field. For further general literature on the study of toposes, consult "Sheaves in Geometry and Logic" by Mac Lane and Moerdijk [MM12], "Elementary Categories, Elementary Toposes" by McLarty [McL92], or for more advanced details Johnstone's "Topos Theory" [Joh14] along with the "Sketches of an Elephant" [Joh02]. Shorter introductions worthy of recommendation are Tom Leinster's "An informal introduction to topos theory" [Lei11], John Baez's "Topos Theory in a Nutshell" [Bae21] or the draft of Kostecki's "An Introduction to Topos Theory" [Kos11].

Elementary Topos

Elementary toposes go back to Lawvere and Tierney [Law69; Law70]. They are not to be confused with "Grothendieck toposes," which originated in the early 1960s in the field of algebraic geometry, and serve as generalisations of topoligical spaces. All elementary toposes are Grothendieck Toposes, but the converse is not necessarily the case. Further details on the history of elementary toposes and their logical aspects are elaborated on by Marquis and Reyes [MR12, Section 3, p. 716ff.].

The definition of an (elementary) topos is as follows:

Definition 2.3. A category $\mathscr{E}$ is an topos if,

1. $\mathscr{E}$ has all finite limits;
2. $\mathscr{E}$ is Cartesian closed (Definition 2.1); and
3. $\mathscr{E}$ has a subobject classifier, i.e. an object Ω and a morphism $\mathrm{true}\colon 1 \rightarrowtail \Omega$ such that for each monomorphism $m\colon S \rightarrowtail B$ in $\mathscr{E}$, there is a unique characteristic morphism $\phi_m\colon B \longrightarrow \Omega$

making the following diagram a pullback-square:

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle \text{true}} \\ B & \xrightarrow[\text{---}]{\phi_m} & \Omega \end{array} \tag{1}$$

For the sake of consistency, we will denote toposes by $\mathscr{E}$, and arbitrary categories by $\mathscr{C}$.

Definition 2.4. We call a morphism $e\colon 1 \rightarrowtail A$, for some object $A \in \mathsf{Ob}(\mathscr{E})$ a *global element* of A. In **Sets**, we understand this as the function $* \mapsto a$ that picks out a single element $a \in A$. Depending on the choice of category (commonly in sheaf categories), these do not exist in general, and we have to use *generalised elements* $X \longrightarrow A$ that denote "elements over X".

As mentioned in the introduction, a topos has sufficient structure to model concepts of set theory. A construct of particular relevance for us are power sets:

Definition 2.5. The power object $\mathbf{P}\,B$ of any object B is such, that for an arbitrary $f\colon B \times A \longrightarrow \Omega$, there exists a unique $g\colon A \longrightarrow \mathbf{P}\,B$ such that the following commutes:

$$\begin{array}{ccccc} A & & B \times A & & \\ {\scriptstyle g}\downarrow & & {\scriptstyle \mathrm{id}_B \times g}\downarrow & \searrow^{f} & \\ \mathbf{P}\,B & & B \times \mathbf{P}\,B & \xrightarrow[\in_B]{} & \Omega \end{array} \tag{2}$$

In a category with exponentials and a subobject classifier Ω (such as a topos), the power object $\mathbf{P}\,B$ is isomorphic to Ω^B. In fact, it is easy to see that the above diagram looks similar to that of exponential objects, and that $\in_B$ is just a more concrete instance of the evaluation morphism $\mathrm{ev}_{B,\Omega}\colon B \times \Omega^B \longrightarrow \Omega$.

Definition 2.6. A subobject is an isomorphism class of monomorphisms, meaning that for two monos $m\colon S \rightarrowtail B$ and $m'\colon S' \rightarrowtail B$, the morphism $f\colon S \longrightarrow S'$ is an isomorphism. The collection of all subobjects are denoted by $\mathrm{Sub}_{\mathscr{C}}(B)$. In a topos $\mathscr{E}$,

$$\mathrm{Sub}_{\mathscr{E}}(A) \cong \mathrm{Hom}_{\mathscr{E}}(A, \Omega) \tag{3}$$

holds for any object $A \in \mathsf{Ob}(\mathscr{E})$ [MM12, Sec. IV.1].

For the above monos m and m' the existence of a non-iso morphism $f\colon S \longrightarrow S'$ such that $m' \circ f = m$ induces a partial order on the subobjects of B.

Remark 2.7 (Homoousion of Subobjects). The following characterisations of a subobject are equivalent:

Monomorphism $m\colon S \rightarrowtail B$, as discussed above, m is a representative element of the equivalence class of monos that constitute the subobject,

Characteristic morphism $\phi_m\colon B \longrightarrow \Omega$, as mentioned in Definition 2.3, the characteristic morphism ϕ_m of any mono m is such that the pullback square $\phi_m \circ m = \mathrm{true} \circ\, !$ commutes.

Global element $s\colon 1 \longrightarrow \mathbf{P}\,B$, it is easy to see that by exponential transposition $\phi\colon B \times 1 \rightarrowtail \Omega$ to $1 \rightarrowtail \Omega^B \cong \mathbf{P}\,B$. In the set-like interpretation, we read this as the morphism that "picks out" a subobject S of B.

Definition 2.8. A category $\mathscr{C}$ has *(epi, mono)-factorisation* when every arrow $f\colon A \longrightarrow B$ factors as $f = m \circ e$, where $e\colon A \twoheadrightarrow B'$ is a epi and $m\colon B' \rightarrowtail B$ is a mono. We refer to the subobject represented by m as the *image* $\operatorname{Im} f$ of f.

Definition 2.9. A (covariant) power object functor $\mathbf{P}(-) : \mathscr{E} \longrightarrow \mathscr{E}$ maps each object B to $\mathbf{P}\,B$. A morphism $f\colon A \longrightarrow B$ is mapped to $\mathbf{P}\,f\colon \mathbf{P}\,A \longrightarrow \mathbf{P}\,B$, defined by the universal property in Equation (2),

$$\begin{array}{ccc} \mathbf{P}\,A & & B \times \mathbf{P}\,A \\ {\scriptstyle \mathbf{P}\,f}\downarrow & & {\scriptstyle \mathrm{id}_B \times \mathbf{P}\,f}\downarrow \quad \searrow^{g} \\ \mathbf{P}\,B & & B \times \mathbf{P}\,B \xrightarrow[\in_B]{} \Omega \end{array} \tag{4}$$

To construct g, first take any subobject X of the form $m\colon X \rightarrowtail A \times \mathbf{P}\,A$, and consider the corresponding characteristic morphism $\phi_m\colon A \times \mathbf{P}\,A \rightarrowtail \Omega$. From this point, we can extend the subobject X by

$$X \xrightarrow{m} A \times \mathbf{P}\,A \xrightarrow{f \times \mathrm{id}_{\mathbf{P}\,A}} B \times \mathbf{P}\,A.$$

The respective characteristic morphism of the new subobject

$$B \times \mathbf{P}\,A \xrightarrow{\phi_{(f \times \mathrm{id}_{\mathbf{P}\,A}) \circ m}} \Omega$$

almost takes the necessary form to serve as g: This only works without further issues if f is itself mono. In general, we have to instead consider the image of $\operatorname{Im}((f \times \mathrm{id}_{\mathbf{P}\,A}) \circ m)$, and its respective characteristic morphism $\phi_{\operatorname{Im}((f \times \mathrm{id}_{\mathbf{P}\,A}) \circ m)}$. For details, consult the Elephant [Joh02, A 2.3].

Example 2.10. The following are examples of toposes:

Category of sets (Sets) It is well known that **Sets** is finitely complete and has exponential objects (sets of functions). The subobject classifier is the two-element set $\Omega \cong \mathbf{2} \cong \{\top, \bot\}$. To verify that this is the subobject classifier, the following pullback square would have to commute:

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & \{\top\} \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{true}} \\ B & \xrightarrow{\phi_m} & \{\top, \bot\} \end{array} \tag{5}$$

where $\mathrm{true} = \top \mapsto \top\colon 1 \rightarrowtail \Omega$ and the characteristic morphism of the mono $m\colon S \rightarrowtail B$ is

$$\phi_m(b\colon B) = \begin{cases} \top & \text{if } \exists\, s \in S.\, m(s) = b \\ \bot & \text{otherwise} \end{cases} \tag{6}$$

Category of finite sets (FinSets) For finite sets, the subobject classifier remains the same, and the above constructions are likewise valid. All finite limits and exponentials exists as well.

Category of Presheaves (Sets$^{\mathscr{C}}$, where $\mathscr{C}$ is small) This functor category is an interesting example that doesn't immediately appear to be "set-like". We do not concern ourselves with the precise definition here, but highlight the definition by Borceux [Bor94, Ex. 5.2.5, p.

295] of the subobject classifier in $\mathbf{Sets}^{\mathscr{C}}$: Ω is a functor, defined for any object $C \in \mathsf{Ob}(\mathscr{C})$ as $\Omega(C)$, the set of all subfunctors of $\mathrm{Hom}_{\mathscr{C}}(-, C)$, and for morphisms $f : D \longrightarrow C$ by the pullback diagram

$$\begin{array}{ccc} \Omega(f)(S) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{C}}(-, D) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(-, f)} & \mathrm{Hom}_{\mathscr{C}}(-, C) \end{array} \tag{7}$$

for all $S \in \Omega(C)$. In any case, it clearly demonstrates that the set-like properties of a topos can take form in ways that differ significantly from the notional equivalent of a two-element set.

(Non-example:) Category of topological spaces (Top) While it is the case that the Sierpinski space over $\{0, 1\}$ would suffice as a subobject classifier [nLa24, Sec. 2.2], the fact that **Top** is not Cartesian closed is a sufficient condition to demonstrate that **Top** is not a topos.

Remark 2.11. In Definition 2.3 a topos was defined as a finitely complete CCC with a subobject classifier. This is the canonical definition advanced by Mac Lane and Moerdijk [MM12, p. IV.1], Lambek and Scott [LS88, Def. 5.4.1, p. 339] Barr and Wells [BW00, p. 2.1], Leinster [Lei11, p. 5], Johnstone [Joh02, Def. 2.1.1], Caramello [Car18, Def. 1.3.28 (a)], Borceux [Bor94, Def. 5.1.3] and Freyd and Scedrov [FS90, p. 1.9].

Bell [Bel08, p. 60] defines a topos as a finitely complete category with a power object functor. From this, it is still possible to derive all exponentials and a subobject classifier.

Historically Lawvere [Law70] and Goldblatt [Gol14, p. 4.3] required a topos to be finitely cocomplete as well. Mikkelsen [Mik76, Thm. 2.3] showed how finite cocompleteness could be derived from the above canonical definition, making the additional requirement redundant.

2.2 Internal Logic of Categories

We now have the means to discuss how to make use of the "set-like" structure of a topos. To put it simply, a topos can serve as a model for intuitionistic, finitist set theory. This fact allows us to translate categorical statements, into the language of set theory in order to simplify definitions and proofs. For instance, recall Remark 2.7: The internal logic allows us to translate between these representations of subobjects $m : S \rightarrowtail B$. Given a mono, we can define that characteristic morphism internally as

$$\phi_m(s) = (s \in B) : S \longrightarrow \Omega,$$

and use that in turn to define the subobject

$$\{\, b : B \mid \phi_m(b) \,\} : \mathbf{P}\, B.$$

The intention here is not to prove that reasoning about categories in the internal logic is sound or complete. We take this as a given and refer to introductory literature on the topic: Mac Lane [Mac13, Sec. IV.5 and 6] introduce "Mitchell-Bénabou Language" and "Kripke-Joyal Semantics" that give an interpretation of set theoretical statements in category theory. Borceux's rigorous enumeration of the inference rules for intuitionistic propositional [Bor94, Sec. 6.7, p. 395] and predicate calculus [Bor94, Sec. 6.8, p. 400], as well as intuitionistic set theory [Bor94, Sec. 6.9, p. 409] are also useful.

The more structure a topos has, the more expressive its internal logic is. For instance, the internal logic of *boolean toposes* [Joh14, Prop. 5.14, p. 138] is not intuitionistic, but classical logic. As an intermediate, we consider categories with less structure and their internal logic.

$$\dfrac{\dfrac{\dfrac{\phi \vdash \phi}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \phi} \quad \dfrac{(\phi \to \psi) \vdash (\phi \to \psi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\phi \to \psi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \psi} \quad \dfrac{\dfrac{\dfrac{\phi \vdash \phi}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \phi} \quad \dfrac{(\phi \to \psi) \vdash (\phi \to \psi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\phi \to \psi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \psi} \quad \dfrac{(\psi \to \chi) \vdash (\psi \to \chi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\psi \to \chi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \chi}}{\dfrac{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\psi \wedge \chi)}{\vdash (\phi \to \psi) \to (\psi \to \chi) \to \phi \to (\psi \wedge \chi)}}$$

Figure 1: A simplified sequent-style natural deduction proof.

The Internal Language of a CCC

The Curry-Howard Correspondence says that a proposition can be derived in (minimal) intuitionistic logic iff a corresponding type can be inhabited in the simply-typed λ-calculus (STLC) [SU06]. This was extended by Lambek [Lam86] to category theory.

Take the intuitionistic tautology

$$\vdash (\phi \to \psi) \to (\psi \to \chi) \to \phi \to (\psi \wedge \chi).$$

In Figure 1 we see a sequent-style natural deduction proof for the proposition. In STLC, we can inhabit the corresponding type

$$\vdash (A \to B) \to (B \to C) \to A \to (B \times C)$$

by the term

$$\vdash \lambda\, f.\, \lambda\, g.\, \lambda\, a.\, (fa), (g(fa)).$$

On closer inspection, a resemblance is found between the lambda-term and the proof in Figure 1: The proof of a conjunction corresponds to the construction of a pair, modus ponens corresponds to λ-abstraction applications, the management of a sequent context corresponds to variable bindings.

The categorical analogue to the above example is to show that the Hom-set

$$\mathrm{Hom}_{\mathscr{C}}\left(1, \left(\left((B \times C)^{A}\right)^{B^{C}}\right)^{B^{A}}\right)$$

is not empty, where $\mathscr{C}$ is a CCC. We interpret the morphism as a map from a context (where 1 signifies the empty context) to a proposition. Exponential objects B^A represent implication and products $A \times B$ represent conjunctions.

To prove this in categorical terms, for an arbitrary CCC we have to provide a morphism with the right domain and codomain. This has to constructed via the morphisms that the structure of our category provides (morphism composition, products, exponentials, ...). In our example, we can show this to be the case by

$$\lambda\lambda \left\langle \mathrm{ev}_{A,B} \circ \langle \pi_3, \pi_1 \rangle, \mathrm{ev}_{B,C} \circ \left\langle \mathrm{ev}_{A,B} \circ \langle \pi_3, \pi_1 \rangle, \mathrm{ev}_{A,C^B} \circ \langle \pi_2, \pi_1 \rangle \right\rangle \right\rangle$$

where $\lambda f \colon A \longrightarrow C^B$ is the transpose of $f \colon A \times B \longrightarrow C$.

We would refer to this approach of demonstrating the existence of a morphism as "external" to $\mathscr{C}$. Contrast this to the "internal" approach: If we can type an analogous type in STLC or prove a logical proposition, then we know that a morphism exists as well.

The Internal Language of a Topos

Mirroring the jump from Definition 2.1 (CCC) to Definition 2.3 (Toposes), the additional structure grants a more expressive "internal language". Crucially, the existence of a subobject classifier gives us the means of talking about formulae in our internal logic.

Recall the definition of a Heyting algebra (a lattice with a greatest and least element, and all implication objects), and that a Heyting algebra can serve as a model for intiutionistic, propositional logic. As explain by Borceux [Bor94, Prop. 6.2.2, p. 349], $\mathrm{Sub}_{\mathscr{E}}(\Omega)$ forms a Heyting algebra. To give an example, conjunction of two elements has the form $\wedge\colon \Omega \times \Omega \longrightarrow \Omega$ and is the Kronecker delta or the characteristic morphism of the diagonal morphism $\Delta_\Omega\colon \Omega \rightarrowtail \Omega \times \Omega$. This means that the conventional connectives $\wedge$, $\vee$, $\implies$, $\neg$ are at our disposal, and even enjoy the inference rules familiar from intuitionistic propositional calculus.

Beyond this, it is also possible to construct existential and universal quantification, as the left and right adjoints of

$$f^*\colon \mathrm{Sub}_{\mathscr{E}}(\Omega) \longrightarrow \mathrm{Sub}_{\mathscr{E}}(1),$$

respectively which in turn is induced by pulling back along $\mathrm{true}\colon 1 \rightarrowtail \Omega$. These too behave in the expected way.

As in Section 2.2, we can define morphisms in the internal logic. For instance, we can define the subobject of the (epi, mono)-split of some morphism $f\colon A \longrightarrow B$ using conventional set-comprehension notation in the same way one would define the image of a function in set theory:

$$\{\, b\colon B \mid \exists\, a\colon A.\, f(a) = b \,\}.$$

Note that the type of this object is $\mathbf{P}\,B \cong \Omega^B$. And in fact, it does bear similarities to a λ-term from B to Ω, as b is a bound variable. To be more accurate though, we have to keep in mind that f is not bound in the above, but we can recognise it to be a free variable/function of type $f\colon A \longrightarrow B$ as mentioned above. The type of the above term should therefore instead be

$$B^A \longrightarrow \mathbf{P}\,B.$$

This is the case for unbound variables in general. By exponential transposition, we can bind the variable as an argument to a function:

$$\mathrm{Im}(f) \coloneqq \{\, b\colon B \mid \exists\, a\colon A.\, f(a) = b \,\},$$

which is now of type $1 \rightarrowtail (\mathbf{P}\,B)^{B^A}$.

The function-like aspects of subobject arise again when considering elementhood. Recall that in Definition 2.5, we noted that $\in_B$ and $\mathrm{ev}_{B,\Omega}$ appear to be similar. This becomes apparent again in the internal logic, as can be seen in this simple example:

$$\begin{aligned}
&\top \in \mathrm{Im}(\mathrm{true}) \\
\iff &\top \in \{\, \omega\colon \Omega \mid \exists\, x\colon 1.\, \mathrm{true}(x) = \omega \,\} && \text{expanding the definition of Im} \\
\iff &\exists\, x\colon 1.\, \mathrm{true}(x) = \top && \text{“applying” } \top \text{ to the subobject} \\
\iff &\mathrm{true}(*) = \top && \text{taking } x = * \text{ as witness} \\
\iff &\top = \top && \text{expanding the definition of true}
\end{aligned}$$

It is not only $\mathrm{Sub}_{\mathscr{E}}(\Omega)$ that forms an (internal) Heyting algebra, but any collection of subobjects. For instance, it is straightforward to define the intersection of two subobjects $m\colon S \rightarrowtail B$ and $m'\colon S' \rightarrowtail B$ as

$$S \cap S' \coloneqq \{\, b\colon B \mid \phi_m(b) \wedge \phi_{m'}(b) \,\}$$

where ϕ_m and $\phi_{m'}$ are respectively the characteristic morphisms of the subobject m and m' given by the universal property of the subobject classifier. Externally, we are constructing the subobject $m_{S \cap S'} : S \cap S' \rightarrowtail B$ of the intersection as the pullback of m_S and $m_{S'}$:

$$\begin{array}{ccc} S \cap S' & \longrightarrow & S' \\ \downarrow & \searrow^{m_{S \cap S'}} & \downarrow m_{S'} \\ S & \xrightarrow{m_S} & B \end{array} \tag{8}$$

This is indicative of a point that we have to keep in mind when dealing with the internal logic of a topos. As we only have finite limits, we cannot speak of arbitrary intersections of a non-finite family of sets, as this would require the non-finite limits.

Example: Naturality of η

We give a internal definition of a natural transformation $\eta \colon \mathrm{Id} \Rightarrow \mathbf{P}$, reminiscent of the unit operation of a monad, and prove it to be natural in the internal logic of $\mathscr{E}$. We will use the fact that natural transformations can be defined and proved to be natural component-wise:

$$\eta_X(x) = \{\, x' \mid x' = x \,\} = \{x\} \tag{9}$$

To verify that this is natural, we first have to give an internal definition of the power object functor (Definition 2.9):

$$\begin{aligned} \mathbf{P}\,(f)\,(S) &\coloneqq \{\, b \colon B \mid \exists\, a \colon S.\, a \in S \wedge b = f(a) \,\} \\ &= \{\, b \colon B \mid \exists\, a \in S.\, b = f(a) \,\}\,. \end{aligned} \tag{10}$$

Proposition 2.12. *η is a natural transformation.*

Proof. We translate the component-wise commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathbf{P}\,A \\ f \downarrow & & \downarrow \mathbf{P}\,f \\ B & \xrightarrow{\eta_B} & \mathbf{P}\,B \end{array}$$

directly into the internal logic of $\mathscr{E}$ as the statement:

$$f \colon A \longrightarrow B \vdash \mathbf{P}\,f \circ \eta_A = \eta_B \circ f.$$

Toposes are functionally extensional: Functions are equal if they agree on all arguments. Hence,

$$f \colon A \longrightarrow B, a \colon A \vdash (\mathbf{P}\,f)(\eta_A(a)) = \eta_B(f(a)).$$

Equality of subobject is similar, in that subobjects are determined by their elements:

$$f \colon A \longrightarrow B, a \colon A, b \colon B \vdash b \in (\mathbf{P}\,f)(\eta_A(a)) \iff b \in \eta_B(f(a)).$$

To continue, we have to expand the internal definitions given in Equations (9) and (10). Assuming the above context, we get the following chain of equations to demonstrate the equality:

$$\begin{aligned}
&b \in \{\, b' \colon B \mid b' = f(a)\,\} \\
\iff\ &b = f(a) \\
\overset{(*)}{\iff}\ &\exists\, a' \colon A.\, a' = a \wedge b = f(a') \\
\iff\ &b \in \{\, b' \colon B \mid \exists\, a' \colon A.\, a' = a \wedge b' = f(a')\,\} \\
\iff\ &b \in \{\, b' \colon B \mid \exists\, a' \colon A.\, a' \in \{a\} \wedge b' = f(a')\,\} \\
\iff\ &b \in \{\, b' \colon B \mid \exists\, a' \colon A.\, a' \in \eta_A(a) \wedge b' = f(a')\,\} \\
\iff\ &b \in (\mathbf{P}\, f)(\eta_A(a))
\end{aligned}$$

In case there is any doubt in the $(*)$ step, we can make use of the fact that the internal logic of an arbitrary topos is intuitionistic. This means that we can use an intuitionistic proof assistant like Coq[1] to convince ourselves that the aforementioned inference holds:

```
Parameter f : Set -> Set.
Parameter a b : Set.

Goal b = f a <-> (exists a', a' = a /\ b = f a').
Proof.
  split.
  - intro H.
    exists a.
    split.
    + congruence.
    + assumption.
  - intro H.
    destruct H as [a' [H1 H2]].
    rewrite H1 in H2.
    assumption.
Qed.
```

or aptly `now firstorder subst.` if there were a need for brevity. ■

Example: Extension of a Function along a Monoid

As a second example, we shall foreshadow coming developments in Chapter 3 (the following definition originates in Carboni, Lack and Walters [CLW93]):

Definition 2.13. A category $\mathscr{C}$ is countably extensive if it exhibits countable coproducts.

We can now define an operation on these countable coproducts:

[1] `https://coq.inria.fr/`, though note that in the Calculus of Constructions certain principles we would like to assume in the internal logic of $\mathscr{E}$ such as functional extensionality do not hold without stating these as `Axioms`.

Definition 2.14. For any function $f \colon B \times A \longrightarrow \mathbf{P}\,B$, the *extension* $\overline{f(-)} \colon B \longrightarrow (\mathbf{P}\,B)^{A^\star}$ iterates over the free monoid A^* successively applying all elements of $\mathbf{P}\,B$ to f, accumulating the results.

Proposition 2.15. *The extension of f is well behaved in the expected fashion.*

Proof. We propose the internal definition of $\overline{f(-)}$ to be

$$\overline{f(b)}(w) = \begin{cases} \{b\} & \text{if } w = \epsilon \\ \bigcup_{b' \in f(b,s)} \overline{f(b')}(w') & \text{if } w = sw' \end{cases} . \tag{11}$$

To show that the definition is "well-behaved," we prove that it is the only definition, as $\overline{f(-)}$ is exactly the initial algebra morphism of the functor $FX = 1 + A \times X$. This means that the following diagram should commute:

$$\begin{array}{ccc} 1 + A \times A^* & \xrightarrow{[\mathsf{nil},\mathsf{cons}]} & A^* \\ {\scriptstyle \mathrm{id}_1 + \mathrm{id}_A \times \overline{f(-)}} \downarrow & & \downarrow {\scriptstyle \overline{f(-)}} \\ 1 + A \times \mathbf{P}\,(B)^B & \xrightarrow{[n,c]} & \mathbf{P}\,(B)^B \end{array} \tag{12}$$

where

$$n(*) = \lambda\, b.\, \{b\}, \tag{13}$$
$$c(s,g) = \lambda\, b. \bigcup_{b' \in f(b,s)} g(b'). \tag{14}$$

We proceed by considering by case distinction on $x \in 1 + A \times A^*$:

1. If $x = \iota_1 *$ (i.e. empty input), then

   $$(\overline{f(-)} \circ \mathsf{nil})(*) = ((\lambda\, a.\, b \mapsto \{b\}) \circ \mathrm{id}_1)(*)$$

   holds trivially.

2. For a non-empty word where $x = \iota_2(s,w)$ and $b \colon B$ arbitrary, consider

   $$\begin{aligned} & \overline{f(-)}(\mathsf{cons}(s,w)) \\ &= \lambda\, b. \bigcup_{b' \in f(b,s)} \overline{f(b')}(w) && \text{by Equation (11)} \\ &= \lambda\, b. \bigcup_{b' \in f(b,s)} \overline{f(-)}(w)(b') \\ &= c(s, \overline{f(-)}(w)) && \text{by Equation (11)} \\ &= (c \circ (\mathrm{id}_A \times \overline{f(-)}))(s,w) && \text{by Equation (14)} \end{aligned}$$

Due to initiality of A^*, we know that there can only be a single morphism that satisfies this property, which is the case for the definition given in Equation (11). ■

2.3 Nondeterministic Automata in a Category

Readers not familiar with the standard construction of a nondeterministic finite automaton (NFA) should consult standard literature like Hopcroft, Motwani and Ullman [HMU06, Sec. 2.3.2, p. 57]. In the following we will consider nondeterministic automata (NDA), generalising NFAs over arbitrary, possibly infinite state spaces. We now come to the definition of NDAs inside a category, as presented by Frank, Milius and Urbat [FMU23, Sec. 6, p. 10]:

Definition 2.16. In an arbitrary category $\mathscr{C}$ with finite limits and (epi, mono)-factorisations, the categorification of NDA, or $\mathscr{C}$-automaton, is given by $A = (Q, \Sigma, \delta, I, F)$

- an object $Q \in \mathsf{Ob}(\mathscr{C})$ of states,
- an object $\Sigma \in \mathsf{Ob}(\mathscr{C})$ representing the input alphabet,
- a subobject $m_\delta \colon \delta \rightarrowtail Q \times \Sigma \times Q$, representing a ternary relation of legal state transitions,
- a subobject $m_I \colon I \rightarrowtail Q$, representing the initial states,
- a subobject $m_F \colon F \rightarrowtail Q$, representing the accepting states.

Remark 2.17. As soon as we recognise that the subobject δ is equivalent to a function of type $Q \times \Sigma \longrightarrow \wp(Q)$ (answering the question "fixing the first two elements of the triple, what are the occurrences of the last Q?"), it is easy to see that for $\mathscr{C} = \mathbf{Sets}$ we get NDAs and that $\mathscr{C} = \mathbf{FinSets}$ gives us NFAs.

Note that an arbitrary topos has all the necessary structure to construct a $\mathscr{C}$-automaton, but that it is not necessary for $\mathscr{C}$ to be a topos.

The semantics of a particular automaton A are the words it will accept, i.e. the "accepting runs" that start in an initial state and after processing a sequence of symbols $s_1 s_2 \dots s_n$ from the input, terminating in an accepting state.

We only assume that $\mathscr{C}$ has finite limits, as to be able to represent words of finite length. We do not assume arbitrary colimits, let alone that $\mathscr{C}$ is countably extensive (Definition 2.13). Therefore, there need not exist a single object that can contain all words $w = s_1 s_2 \dots s_n$ of arbitrary length.

For that reason, the accepted runs are defined as a family of subobjects for each length of the accepting run:

$$L \coloneqq \left(m_n^{(L)} \colon L^{(n)} \rightarrowtail \Sigma^n \right)_{n \in \mathbb{N}}, \tag{15}$$

Each constituent of the family is respectively defined in terms of the commutative diagrams in Figure 2 and Figure 3:

Accepted words of length $n = 0$ There is only a single accepted word of nil-length, ϵ (the empty word). An automaton accepts this iff there is an initial state that is also accepting. Categorically, this corresponds to the pullback of the subobjects m_I and m_F.

It is therefore not surprising to note that $\Sigma^0 \cong 1$ has two subobjects: the terminal object itself and the initial object (by definition of initiality in the Heyting algebra).

If $I \cap F$ is an "empty intersection", in which case it is isomorphic to 0, then the image $\mathrm{Im}(!) = L^{(0)}(A)$ of $0 \longrightarrow 1$ is likewise 0. Otherwise, the image denotes the singleton subobject $\{\epsilon\}$. This is a satisfactory definition of $m_0^{(L)} \colon L^{(0)} \rightarrowtail \Sigma^0$ and intuitively matches our expectations from automata theory.

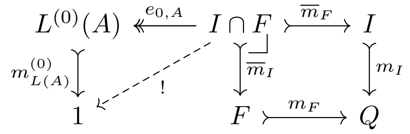

Figure 2: Commutative diagrams describing a NDA for $n = 0$

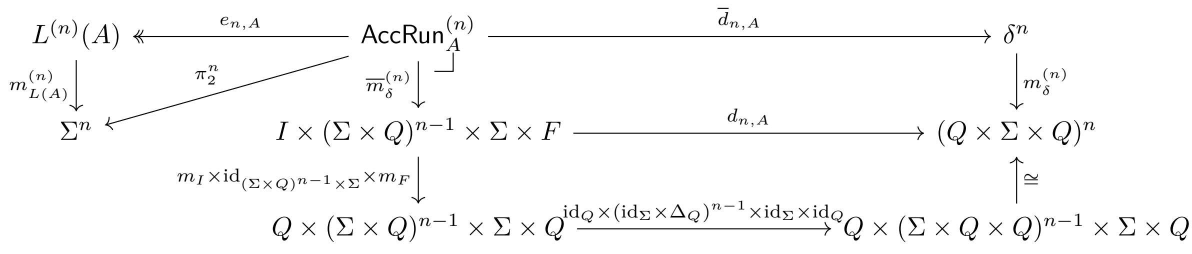

Figure 3: Commutative diagrams describing a NDA for $n \geq 1$

Accepted words of length $n \geq 1$ For any non-empty word $w = s_1 s_2 \dots s_n$, we need to ensure that there is a legal "accepting run", i.e. for a subobject of δ^n, the right state of the entry i matches the left state of the entry $i + 1$. In Figure 3 we describe the accepting runs by a pullback of the subobject $m_\delta^{(n)} \colon \delta^n \rightarrowtail (Q \times \Sigma \times Q)^n$ of n arbitrary legal transitions and the morphism $d_{n,A}$ that reorders a

$$q_1, ((s_1, q_2), (s_2, s_3), \dots, (s_{n-2}, q_{n-1}), (s_{n-1}, q_n)), s_n, q_{n+1}$$

where $q_1 \in I$ and $q_{n+1} \in F$, by duplicating each mid-state

$$q_1, s_1, q_2, q_2, s_2, q_3, \dots, s_{n-2}, q_{n-1}, s_{n-1}, q_n, q_n, s_n, q_{n+1}$$

and then utilising the associativity of products to re-order the these into the intended form:

$$(q_1, s_1, q_2), (q_2, s_3, q_3) \dots, (q_{n-1}, s_{n-1}, q_n), (q_n, s_n, q_{n+1}),$$

thus appearing as a subobject of $(Q \times \Sigma \times Q)^n$.

The accepted words are of course a subobject of $m_{L(A)}^{(n)} \colon L^{(n)}(A) \rightarrowtail \Sigma^n$, that correspond to the image monomorphism given by the (epi,mono)-factorisation of the morphism $\pi_2^n \colon \mathsf{AccRun}_A^{(n)} \longrightarrow \Sigma^n$ projecting the symbols in the input alphabet that constitute the accepting run.

3 $\mathscr{C}$-Automata and Coalgebras

Readers unfamiliar with F-coalgebras (sometimes also referred to as "coalgebras over an endofunctor"), can consult the introduction by Jacobs [Jac17].

Recall that a general F-coalgebra is specified by an endofunctor F over a category $\mathscr{C}$. In the following we will consider special cases of F, defined by composing an arbitrary functor G with a monad T. We follow and recapitulate the results of Jacobs, Silva and Sokolova [JSS12] to demonstrate how and when this ensures the existence of a terminal coalgebra.

3.1 Eilenberg-Moore Algebras and their Semantics

Definition 3.1. An *Eilenberg-Moore algebra* over a monad (T, μ, η) is a morphism $\alpha\colon TX \longrightarrow X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & \searrow\!\!\!\!= & \downarrow \alpha \\ & & X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{\mu_X} & TX \\ {\scriptstyle T(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TX & \xrightarrow{\alpha} & X \end{array} \tag{16}$$

both commute.

Definition 3.2. The *category of Eilenberg-Moore Algebras* $\mathcal{EM}(T)$ is a restriction of the category of T-algebras $\mathbf{Alg}(T)$ to objects that satisfy Equation (16). Notions such as initial algebras in $\mathcal{EM}(T)$ are analogous to those in $\mathbf{Alg}(T)$.

Definition 3.3. For a monad $(T\colon \mathscr{C} \longrightarrow \mathscr{C}, \mu, \eta)$ and an arbitrary endofunctor $G\colon \mathscr{C} \longrightarrow \mathscr{C}$, a *distributive* $\mathcal{EM}$*-law* is a natural transformation

$$\rho\colon TG \Rightarrow GT, \tag{17}$$

such that the following two commutative diagrams commute:

$$\begin{array}{ccc} & GX & \\ {\scriptstyle \eta_{GX}}\swarrow & & \searrow{\scriptstyle G(\eta_X)} \\ TGX & \xrightarrow{\rho_X} & GTX \end{array} \qquad \begin{array}{ccccc} TTGX & \xrightarrow{T(\rho_X)} & TGTX & \xrightarrow{\rho_{TX}} & GTTX \\ \downarrow{\scriptstyle \mu_{GX}} & & & & \downarrow{\scriptstyle G(\mu_X)} \\ TGX & & \xrightarrow{\rho_X} & & GTX \end{array} \tag{18}$$

Definition 3.4. Any functor $G\colon \mathscr{C} \longrightarrow \mathscr{C}$ can be *lifted* from $\mathscr{C}$ to an Eilenberg-Moore category as $\hat{G}\colon \mathcal{EM}(T) \longrightarrow \mathcal{EM}(T)$, given a $\mathcal{EM}$-law $\rho\colon TG \Rightarrow GT$.

The object map of $\hat{G}$ is defined as

$$\left(TX \xrightarrow{\alpha} X\right) \mapsto \left(TGX \xrightarrow{\rho_X} GTX \xrightarrow{G(\alpha)} GX\right) \tag{19}$$

and the morphism map of $\hat{G}$ as $f \mapsto G(f)$.

Jacobs' construction relies on the existence of a final coalgebra (Z, ζ) in $\mathbf{Coalg}(G)$. Given $\zeta\colon Q \longrightarrow GQ$, we can construct another object in $\mathbf{Coalg}(G)$

$$TZ \xrightarrow{T(\zeta)} TGZ \xrightarrow{\rho_Z} GTZ,$$

and know that there must be a unique morphism $\alpha\colon TZ \longrightarrow Z$ from the latter to the former:

$$\begin{array}{ccc} TZ & \xrightarrow{\rho \circ T(\zeta)} & GTZ \\ {\scriptstyle \alpha}\,\vdots\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ Z & \xrightarrow{\zeta} & GZ \end{array} \qquad \text{in } \mathscr{C}$$

Recognising that the morphism ζ constitutes an Eilenberg-Moore algebra, a change of perspective reveals an object in $\mathbf{Coalg}(\hat{G})$:

$$\begin{matrix} TZ \\ \downarrow\alpha \\ Z \end{matrix} \xrightarrow{\zeta} \hat{G}\begin{pmatrix} TZ \\ \downarrow\alpha \\ Z \end{pmatrix} \qquad \text{in } \mathcal{EM}(\mathrm{T})$$

as $\hat{G}(\alpha) = TGZ \xrightarrow{G(\alpha)\circ\rho} GZ$. It is of special interest to us that the lifted morphism ζ is also the final coalgebra in $\mathcal{EM}(T)$ with the same carrier.

Example 3.5 (Semantic Map of a NDA in **Sets**). If we take $GQ = \mathbf{2} \times Q^\Sigma$ and $TQ = \wp(Q)$, the carrier of the final coalgebra is $Z = \wp(\Sigma^\star)$. Given a $\mathcal{EM}$-law that distributes $\wp(-)$ over G and the finality of α, the construction provides a map from an arbitrary state Q to the set of accepted words,

$$[\![-]\!] : Q \xrightarrow{\eta_Q} \mathbf{P}\,Q \xrightarrow{\alpha} \wp(\Sigma^\star)\,.$$

3.2 Topos Semantics of a Coalgebra

To investigate the relation of a coalgebra and a nondeterministic automaton in a topos, we will begin by describing the trace semantics of a coalgebra in an arbitrary topos. The functor of the coalgebra we will be considering is

$$FQ = \Omega \times \mathbf{P}(Q)^\Sigma. \tag{20}$$

We shall proceed by the approach sketched in Section 3.1, regarding Equation (20) as the composition GT of the functors

$$GQ = \Omega \times Q^\Sigma \qquad TQ = \mathbf{P}\,Q$$

where T has the canonical monadic structure (T, μ, η).

Validity of the $\mathcal{EM}$-law in a Topos

Before attempting to construct the semantic map $[\![-]\!] : Q \longrightarrow \wp(\Sigma^\star)$, we have to verify that there exists a natural transformation

$$\rho\colon\ \mathbf{P}\left(\Omega \times -^\Sigma\right) \Rightarrow \Omega \times \mathbf{P}(-)^\Sigma \tag{21}$$

for which the $\mathcal{EM}$-law holds (Definition 3.3). We translate the definition by Jacobs, Silva and Sokolova [JSS12, Sec. 5.1, p. 117] into the internal language of $\mathscr{E}$

$$\begin{aligned} &\rho_Q \coloneqq \langle \rho_{Q,1}, \rho_{Q,2} \rangle \\ &\text{where,} \\ &\rho_{Q,1}(S) \coloneqq \exists\ \langle o, t\rangle :\ \in S.\, o, \\ &\rho_{Q,2}(S) \coloneqq \lambda\, s\colon \Sigma.\ \bigcup_{\langle o,t\rangle \in S} t(s). \end{aligned} \tag{22}$$

Proposition 3.6. *ρ is a natural transformation.*

Proof. We want to show that there is a natural way to "compress" a set of deterministic automata into a single non-deterministic automaton, which is accepting as a whole when a single deterministic automaton would be accepting as well.

Categorically, this corresponds to showing that the following diagram commutes,

$$\begin{array}{ccccc} Q & & \mathbf{P}\left(\Omega \times Q^\Sigma\right) & \xrightarrow{\quad\rho_Q\quad} & \Omega \times \mathbf{P}(Q)^\Sigma \\ \big\downarrow f & & \big\downarrow TF(f) & & \big\downarrow FT(f) \\ Q' & & \mathbf{P}\left(\Omega \times Q'^\Sigma\right) & \xrightarrow{\quad\rho'_Q\quad} & \Omega \times \mathbf{P}(Q')^\Sigma \end{array}$$

or in the internal logic of $\mathscr{E}$

$$S\colon \mathbf{P}\left(\Omega \times Q^{\Sigma}\right), f\colon Q \longrightarrow Q' \vdash \rho_{Q'}(TG(f)(a)) = GT(f)(\rho_Q(a)).$$

We can see that this is given by considering the following chain of equivalences:

$$\begin{aligned}
&\rho_{Q'}(TF(f)(S)) \\
&= \rho_{Q'}\left(\left\{\, (o, f \circ t)\colon \Omega \times Q^{\Sigma} \;\middle|\; (o,t) \in S \,\right\}\right) \\
&= \left\langle \exists\, \langle o,t\rangle \in S.\, o(a), \lambda\, s. \bigcup_{\langle o,t\rangle \in S} f(t(a)(s)) \right\rangle \\
&= \left\langle \exists\, \langle o,t\rangle \in S.\, o(a), T(f) \circ (\lambda\, s. \bigcup_{\langle o,t\rangle \in S} t(s)) \right\rangle \\
&= GT(f)\left(\left\langle \exists\, \langle o,t\rangle \in S.\, o(a), \lambda\, s. \bigcup_{\langle o,t\rangle \in S} t(s) \right\rangle\right) \\
&= GT(f)(\rho_Q(S))
\end{aligned}$$

■

Having established the naturality of ρ, we can directly proceed to verify the properties of the $\mathcal{EM}$-laws (Equation (16)):

Lemma 3.7. *ρ satisfies the unit/distributivity law.*

Proof. Thinking internally, we want to demonstrate that ρ will merge a single, deterministic automaton in the same way as if the deterministic automaton were to always return a singleton successor state.

Diagrammatically, we want to show

$$\begin{array}{ccc}
 & \Omega \times Q^{\Sigma} & \\
\swarrow^{\eta_{GQ}} & & \searrow^{G(\eta_Q)} \\
\mathbf{P}\left(\Omega \times Q^{\Sigma}\right) & \xrightarrow[\rho_Q]{} & \Omega \times \mathbf{P}\, Q^{\Sigma}
\end{array} \qquad \vdash \rho_Q \circ \eta_{GQ} = G(\eta_Q) \tag{23}$$

We prove this to be the case by taking a $\langle o,t\rangle : \Omega \times Q^{\Sigma}$:

$$\begin{aligned}
&\rho_Q(\eta_{FQ}(\langle o,t\rangle)) \\
&= \left\langle \exists\, \langle o',t'\rangle \in \{\langle o,t\rangle\}.\, o', \lambda\, s. \bigcup_{\langle o',t'\rangle \in \{\langle o,t\rangle\}} t'(s) \right\rangle \\
&= \langle o, \lambda\, s. \{t(s)\}\rangle \\
&= \langle o, \eta_Q \circ t\rangle \\
&= \langle \mathrm{id}_\Omega, \eta_Q\rangle\,(\langle o,t\rangle) = G(\eta_Q)(\langle o,t\rangle)
\end{aligned}$$

■

Lemma 3.8. *ρ satisfies the multiplication/associativity law.*

Proof. The claim in our case is that when flattening power objects of deterministic automata, the order in which we flatten these into a single nondeterministic automaton is not relevant with regard to ρ. Formally, the following diagram has to commute:

$$\begin{array}{ccccc} \mathbf{P}\left(\mathbf{P}\left(\Omega \times Q^{\Sigma}\right)\right) & \xrightarrow{\mathbf{P}\,\rho_Q} & \mathbf{P}\left(\Omega \times \mathbf{P}\,Q^{\Sigma}\right) & \xrightarrow{\rho_{\mathbf{P}\,Q}} & \Omega \times \mathbf{P}\left(\mathbf{P}\left(Q\right)\right)^{\Sigma} \\ \downarrow{\scriptstyle \mu_{GQ}} & & & & \downarrow{\scriptstyle G(\mu_Q)} \\ \mathbf{P}\left(\Omega \times Q^{\Sigma}\right) & & \xrightarrow[\rho_Q]{\qquad\qquad} & & \Omega \times \mathbf{P}\left(Q\right)^{\Sigma} \end{array} \tag{24}$$

We can split this up into two equations:

1. The first ends in Ω:

$$\begin{array}{ccccc} \mathbf{P}\left(\mathbf{P}\left(\Omega \times Q^{\Sigma}\right)\right) & \xrightarrow{\mathbf{P}\,\rho_Q} & \mathbf{P}\left(\Omega \times \mathbf{P}\,Q^{\Sigma}\right) & \xrightarrow{\pi_1 \circ \rho_{\mathbf{P}\,Q}} & \Omega \\ \downarrow{\scriptstyle \mu_{GQ}} & & & & \downarrow{\scriptstyle \mathrm{id}_\Omega} \\ \mathbf{P}\left(\Omega \times Q^{\Sigma}\right) & & \xrightarrow[\pi_1 \circ \rho_Q]{\qquad\qquad} & & \Omega \end{array} \tag{25}$$

The internal proof assumes a $S\colon \mathbf{P}\left(\mathbf{P}\left(\Omega \times Q^{\Sigma}\right)\right)$, and proceeds as follows

$$\begin{aligned} & (\mathrm{id}_\Omega \circ \rho_{\mathbf{P}\,Q,1} \circ \mathbf{P}\,\rho_Q)(S) \\ \iff & \rho_{Q,1}(\mathbf{P}\,\rho_{\mathbf{P}\,Q}(S)) \\ \iff & \exists\, \langle o, t\rangle \in \mathbf{P}\,\rho_Q(S).\, o \\ \iff & \exists\, \langle o, t\rangle \in \left\{ a' \colon \Omega \times Q^{\Sigma} \;\middle|\; \exists\, s \in S.\, a' = \rho_Q(s) \right\}.\, o(a) \\ \overset{(*)}{\iff} & \exists\, \langle o, t\rangle \in \left\{ \langle o', t'\rangle : \Omega \times Q^{\Sigma} \;\middle|\; \exists\, s \in S. \in s \right\}.\, o(a) \\ \iff & \rho_{Q,1}(\mu_{GQ}(S)) \iff (\rho_{Q,1} \circ \mu_{GQ})(S) \end{aligned}$$

where $(*)$ holds, as we are only considering the operation on the first component.

2. The second one ends in $\mathbf{P}\left(Q\right)^{\Sigma}$:

$$\begin{array}{ccccc} \mathbf{P}\left(\mathbf{P}\left(\Omega \times Q^{\Sigma}\right)\right) & \xrightarrow{\mathbf{P}\,\rho_Q} & \mathbf{P}\left(\Omega \times \left(\mathbf{P}\,Q\right)^{\Sigma}\right) & \xrightarrow{\pi_2 \circ \rho_{\mathbf{P}\,Q}} & \left(\mathbf{P}\left(\mathbf{P}\left(Q\right)\right)\right)^{\Sigma} \\ \downarrow{\scriptstyle \mu_{GQ}} & & & & \downarrow{\scriptstyle {\mu_Q}^{\Sigma}} \\ \mathbf{P}\left(\Omega \times Q^{\Sigma}\right) & & \xrightarrow[\pi_2 \circ \rho_Q]{\qquad\qquad} & & \left(\mathbf{P}\left(Q\right)\right)^{\Sigma} \end{array} \tag{26}$$

As before, we assume a $S\colon \mathbf{P}\left(\mathbf{P}\left(\Omega \times Q^{\Sigma}\right)\right)$:

$$\begin{aligned} & ({\mu_Q}^{\Sigma} \circ \rho_{\mathbf{P}\,Q,2} \circ \mathbf{P}\,\rho_Q)(S) \\ = {} & \lambda\, s.\, \mu_Q(((\rho_{\mathbf{P}\,Q,2} \circ \mathbf{P}\,\rho_Q)(S))(s)) \\ = {} & \lambda\, s.\, \mu_Q(\left(\lambda\, s. \bigcup_{\langle o,t\rangle \in \mathbf{P}\,\rho_Q} t(s)\right)(s)) \\ = {} & \lambda\, s.\, \mu_Q\left(\bigcup_{\langle o,t\rangle \in \mathbf{P}\,\rho_Q(S)} t(s)\right) \end{aligned}$$

$$
\begin{aligned}
&= \lambda\, s.\, \mu_Q\, (\{\, y\colon \mathbf{P}\, Q \mid \exists\, \langle o,t\rangle \in \mathbf{P}\, \rho_Q(S).\, y = t(s)\,\})\\
&= \lambda\, s.\, \mu_Q \left(\left\{\, y\colon \mathbf{P}\, Q \;\middle|\; \exists\, \langle o,t\rangle \in \left\{\, a'\colon \Omega \times \mathbf{P}\, Q^{\Sigma} \;\middle|\; \exists\, x \in S.\, a' = \rho_Q(x)\,\right\}.\, y = t(s)\,\right\}\right)\\
&= \lambda\, s.\, \mu_Q\, (\{\, y\colon \mathbf{P}\, Q \mid \exists\, x \in S.\, \exists\, \langle o,t\rangle \in \rho_Q(x).\, y = t(s)\,\})\\
&= \lambda\, s.\, \mu_Q \left(\left\{\, y\colon \mathbf{P}\, Q \;\middle|\; \exists\, x \in S.\, \exists\, t \in \left\{\, a \in x \;\middle|\; \lambda\, s. \bigcup_{\langle o,t\rangle \in x} t(s)\,\right\}.\, y = t(s)\,\right\}\right)\\
&= \lambda\, s.\, \mu_Q \left(\left\{\, y\colon \mathbf{P}\, Q \;\middle|\; \exists\, x \in S.\, \exists\, a \in x.\, y = \bigcup_{\langle o,t\rangle \in x} t(a)(s)\,\right\}\right)\\
&= \lambda\, s.\, \{\, q\colon Q \mid \exists\, x \in S.\, \exists\, \langle o,t\rangle \in x.\, q = t(a)(s)\,\}\\
&= \lambda\, s.\, \{\, q\colon Q \mid \exists\, a \in \{\, \langle o,t\rangle : A \mid \exists\, x \in S.\, a' \in x\,\}\,.\, q = t(s)\,\}\\
&= \lambda\, s.\, \{\, q\colon Q \mid \exists\, \langle o,t\rangle \in \mu_{GQ}(S).\, q = t(s)\,\}\\
&= \lambda\, s. \bigcup_{\langle o,t\rangle \in \mu_{GQ}(S)} t(s)\\
&= \rho_{Q,2}(\mu_{GQ}(S)) = (\rho_{Q,2} \circ \mu_{GQ})(S) \qquad \blacksquare
\end{aligned}
$$

Proposition 3.9. *ρ satisfies the $\mathcal{EM}$-law.*

Proof. This follows from Proposition 3.6 and Lemmas 3.7 and 3.8. ■

Terminal Coalgebra Construction

The above results allows us to apply the construction sketched in Section 3.1. If we can provide a final coalgebra for the functor G, then we can infer the existence of a final coalgebra for GT with the same carrier.

The immediate issue is that, when recalling from **Sets** the canonical carrier of the coalgebra for G is $\mathbf{P}\,(\Sigma^*)$, that as a potentially infinite object does not fit into the finitist universe of a topos. In fact, it is difficult to see how we can represent a collection of words with arbitrary lengths with only finite (co-)limits.

We therefore have to strengthen our assumptions, and assume the existence of a Σ^*-like object[2] in the base category, i.e. that $\mathscr{E}$ is countably extensive (see Definition 2.13).

Proposition 3.10. *The functor G has a final coalgebra.*

Proof. The argument proceed analogously to Jacobs [Jac17, Proposition 2.3.5, p. 70] in **Sets**.

As implied above, the carrier of coalgebra on G will be

$$
\mathbf{P}\,(\Sigma^*) \coloneqq \mathbf{P}\left(\coprod_{n\in\mathbb{N}} \Sigma^n\right),
$$

with the expected family projection morphisms $(\pi_n\colon\, \mathbf{P}\,(\Sigma^*) \longrightarrow \mathbf{P}\,(\Sigma^n))_{n\in\mathbb{N}}$.

[2] As discussed by Iwaniack [Iwa24, Prop. 1.22], a topos has a natural number object (NNO) iff it has a free monoid object, which Σ^* ist. So our results can apply in toposes with just a NNO as well. One such example is Hyland's "Effective Topos" [Hyl82], which has an NNO but does not exhibit arbitrary countable coproducts.

We intend to show that for any other coalgebra $\langle \tilde{o}, \tilde{t} \rangle : X \longrightarrow \Omega \times X^{\Sigma}$ there exists a coalgebra homomorphism $h\colon X \longrightarrow \mathbf{P}(\Sigma^*)$, defined internally as

$$h(x\colon X) \coloneqq \left\{ w\colon \Sigma^* \;\middle|\; \bar{o}(\overline{\tilde{t}(x)}(w)) \right\}, \tag{27}$$

where $\overline{\tilde{t}(x)}$ is the extension of $\tilde{t}$ along a free monoid starting with x, as elaborated on in Section 2.2 and $\bar{o}\colon \mathbf{P}\,Q \longrightarrow \Omega$ is defined as the extension of o that checks if any element in a power object of states is accepting

$$\bar{o}(S) \coloneqq \exists\, q \in Q.\, o(x) = \text{true}\,. \tag{28}$$

Next, we can recall the familiar definition of $\langle o, t \rangle$, that we can now state given some countably extensive topos:

$$o(S) = \epsilon \in S \qquad\qquad o\colon\ \mathbf{P}(\Sigma^*) \longrightarrow \Omega, \tag{29}$$

$$t(S) = \lambda\, s.\, \{\, w\colon \Sigma^* \mid sw \in S \,\} \qquad\qquad t\colon\ \mathbf{P}(\Sigma^*) \longrightarrow (\mathbf{P}(\Sigma^*))^{\Sigma} \tag{30}$$

The claim we first want to verify is therefore

$$\langle o, t \rangle \circ h = (\mathrm{id}_{\Omega} \times h^{\Sigma}) \circ \langle \tilde{o}, \tilde{t} \rangle\,,$$

which we establish separately for each side of the product:

- For the left side:

$$\begin{aligned}
& (o \circ h)(x) && \\
\iff\ & o(h(x)) && \\
\iff\ & \epsilon \in \left\{ w\colon \Sigma^* \;\middle|\; \bar{o}(\overline{\tilde{t}(x)}(w)) \right\} && \text{by definition of } h \\
\iff\ & \bar{o}(\overline{\tilde{t}(x)}(\epsilon)) \iff \bar{o}(x) && \text{by definition of } \overline{t(-)} \\
\iff\ & \mathrm{id}_{\Omega}(\tilde{o}(x)) && \\
\iff\ & (\mathrm{id}_{\Omega} \circ \bar{o})(x) && \\
\iff\ & \bar{o}(x) &&
\end{aligned}$$

- For the right side:

$$\begin{aligned}
& (t \circ h)(x) && \\
=\ & t(h(x)) && \\
=\ & \lambda\, s.\, \{\, w\colon \Sigma^* \mid sw \in h(x) \,\} && \\
=\ & \lambda\, s.\, \left\{ w\colon \Sigma^* \;\middle|\; sw \in \left\{ w'\colon \Sigma^* \;\middle|\; \bar{o}(\overline{\tilde{t}(x)}(w')) \right\} \right\} && \text{by definition of } h \\
=\ & \lambda\, s.\, \left\{ w\colon \Sigma^* \;\middle|\; \bar{o}(\overline{\tilde{t}(x)}(sw)) \right\} && \text{by definition of } \in_{\mathbf{P}(\Sigma^*)} \\
=\ & \lambda\, s.\, \left\{ w\colon \Sigma^* \;\middle|\; \bar{o}(\overline{\tilde{t}(t(x)(s))}(w)) \right\} && \text{by definition of } \overline{t(-)} \\
=\ & \lambda\, s.\, h(\tilde{t}(x)(s)) && \text{by definition of } h \\
=\ & h^{\Sigma}(\tilde{t}(x)) = (h^{\Sigma} \circ \tilde{t})(x) && \text{by definition of } -^{\Sigma}
\end{aligned}$$

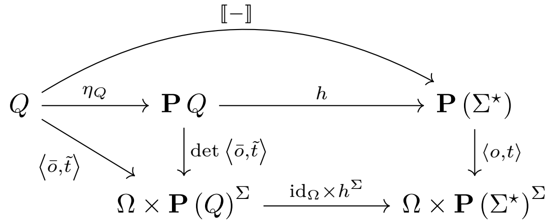

Figure 4: Overview of the construction.

It remains to substantiate that h is unique: Assume a satisfactory $g \colon X \longrightarrow \mathbf{P}(\Sigma^*)$ (satisfying $\langle o, t\rangle \circ g = \langle \mathrm{id}_\Omega, g^\Sigma \rangle \circ \bar{o} \times \tilde{t}$), we want to attempt to equate it to h. Given an arbitrary $x \colon X$ and $w \in \mathbf{P}(\Sigma^*)$, we rephrase the equality as

$$w \in g(x) \iff w \in h(x).$$

Proceeding by induction over the length of w:

Induction Basis ($w = \epsilon$) Can easily seen to be the case:

$$\epsilon \in g(x) \iff o(g(x)) \iff \bar{o}(x) \iff o(h(x)) \iff \epsilon \in h(x).$$

Induction Step ($w = sw'$) By the induction hypothesis we know

$$w' \in h(x) \iff \bar{o}(\overline{\tilde{t}(x)}(w')) \iff w' \in g(x)$$

holds for any x. This allows us to reason:

$$\begin{aligned}
& sw' \in h(x) \\
\iff & sw' \in \left\{ w \;\middle|\; \bar{o}(\overline{\tilde{t}(x)}(w)) \right\} \\
\iff & \bar{o}(\overline{\tilde{t}(x)}(sw')) \\
\iff & \bar{o}(\overline{\tilde{t}(\tilde{t}(s)(x))}(w')) && \text{by definition of Equation (11)} \\
\iff & o(g(\tilde{t}(x)(s))) && \text{as } g \text{ is a homomorphism} \\
\iff & w' \in g(\tilde{t}(x)(s)) \\
\iff & sw' \in g(x)
\end{aligned}$$

■

From this we can conclude that $\mathbf{P}(\Sigma^*)$ is also the carrier of the final coalgebra of $\hat{G}$ in $\mathcal{EM}(T)$. The finality in $\mathcal{EM}(T)$ ensures that there exists a (unique) morphism $h \colon \mathbf{P}\, Q \longrightarrow \mathbf{P}(\Sigma^*)$ in $\mathscr{E}$.

We can use this to construct the semantic map, that sends a state to the accepted words:

$$[\![-]\!]_n \coloneqq h \circ \eta_Q = \left\{ w \;\middle|\; \bar{o}(\overline{t(-)}(w)) \right\}, \tag{31}$$

where the definition of h is:

$$h(S) \coloneqq \left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(w)) \right\}. \tag{32}$$

The uniqueness ensures that this definition of h is correct, if the necessary conditions are satisfied. These are illustrated in Figure 4. As an aside, the "de-determinisation" following Silva et al. [Sil+13, p. 5] of a deterministic automaton, is internally given by

$$\det \langle \bar{o}, \tilde{t} \rangle(S) \coloneqq \left\langle \exists\, q \in S.\, \bar{o}(q), \lambda\, s.\, \bigcup_{x \in S} \tilde{t}(x)(s) \right\rangle. \tag{33}$$

We are primarily interested in verifying that the right square commutes, as the commutativity of the left and the top triangle is easy to see:

$$\begin{aligned} \exists\, q \in \{\tilde{q}\}.\, o(q) &\iff o(\tilde{q}) \\ \lambda\, s.\, \bigcup_{x \in \{\tilde{q}\}} t(x)(s) &= \lambda\, s.\, t(\tilde{q})(s) \end{aligned}$$

As before, we will assume a $S \colon \mathbf{P}\, Q$ and show that the square commutes by considering both sides of the resulting product separately.

1. For the left side,

$$\begin{aligned} & \mathrm{id}_\Omega(\pi_1(\det \langle \bar{o}, \tilde{t} \rangle(S))) \\ \iff & \pi_1(\det \langle \bar{o}, \tilde{t} \rangle(S)) \\ \iff & \exists\, q \in S.\, \bar{o}(q) \\ \iff & \exists\, q \in S.\, \bar{o}(\overline{t(q)}(\epsilon)) && \text{by definition of } \overline{\tilde{t}(-)} \\ \iff & \epsilon \in \left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(w)) \right\} && \text{by definition of } \bar{o} \\ \iff & \bar{o}\left(\left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(w)) \right\} \right) \\ \iff & o(h(S)) \end{aligned}$$

2. For the right side,

$$\begin{aligned} & h^\Sigma(\pi_2(\det \langle \bar{o}, \tilde{t} \rangle(S))) \\ = & \lambda\, s.\, h\left(\bigcup_{x \in S} \tilde{t}(x)(s) \right) \\ = & \lambda\, s.\, \left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in \left(\bigcup_{x \in S} \tilde{t}(x)(s) \right).\, \bar{o}(\overline{t(q)}(w)) \right\} \\ = & \lambda\, s.\, \left\{ w' \colon \Sigma^* \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(sw')) \right\} \\ = & \lambda\, s.\, \left\{ w' \colon \Sigma^* \;\middle|\; sw' \in \left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(w)) \right\} \right\} \\ = & t\left(\left\{ w \colon \Sigma^n \;\middle|\; \exists\, q \in S.\, \bar{o}(\overline{t(q)}(w)) \right\} \right) \\ = & t(h(S)) \end{aligned}$$

This gives us the necessary assurance in that Equation (31) is the internal description of the semantics of the GT-coalgebra.

3.3 Topos Semantics of a $\mathscr{C}$-Automaton

The internalisation of a NDA as a $\mathscr{C}$-automaton (Section 2.3) is straightforward, as a $\mathscr{C}$-automata are defined in such a way that the structure of a topos is sufficient.

We consider the separate cases:

The empty word As mentioned in Section 2.2, we can describe the pullback on the right side of Figure 2 by

$$I \cap F = \{\, q \in Q \mid q \in I \wedge q \in F \,\} \tag{34}$$

As before, the image of the morphism $! \colon I \cap F \longrightarrow 1$ will only contain ϵ when $I \cap F$ is non-empty:

$$L^{(0)}(A) = \begin{cases} \varnothing & \text{if } I \cap F \cong \varnothing, \\ \{\epsilon\} & \text{otherwise.} \end{cases} \tag{35}$$

Non-empty words Again in the case of Figure 3, the pullback can be expressed as

$$\mathsf{AccRun}_A^{(n)} = \mathrm{Pb}\,(d_{n,A}, m_\delta^n) = \{\, r \in (Q \times \Sigma \times Q)^n \mid r \in \mathrm{Im}(d_{n,A}) \wedge r \in \mathrm{Im}(m_\delta^n) \,\} \tag{36}$$

While m_δ^n is a straightforward injection, the image of $d_{n,A}$ is slightly more complicated as it assures that the accepting run is well-formed:

1. The run begins in an initial state,
2. The run ends in an accepting state,
3. Adjacent pairs in a run share the same element in the third and first component respectively, making the chain of transitions legal.

As such, we can define the pullback internally as a subobject of δ^n that satisfies these conditions. Borrowing an idea from Lisp (where for instance the function `cadr` is defined as the composition of `car` and `cdr`), we will abbreviate $\pi_i(\pi_j(x))$ as $\pi_{i,j}(x)$:

$$\mathsf{AccRun}_A^{(n)} = \left\{ a \in \delta^n \;\middle|\; \underbrace{\pi_{1,1}(a) \in I}_{\text{Cond. 1}} \wedge \underbrace{\pi_{3,n}(a) \in F}_{\text{Cond. 2}} \wedge \underbrace{\forall\, 1 \leq i < n.\, \pi_{3,i}(a) = \pi_{1,i+1}(a)}_{\text{Cond. 3}} \right\}.$$

The image of $\mathsf{AccRun}_A^{(n)} \longrightarrow \Sigma^n$, i.e. the language at length n, can be straightforwardly described as

$$L^{(n)}(A) = \left\{ w \in \Sigma^n \;\middle|\; \exists\, a \in \mathsf{AccRun}_A^{(n)}.\, \pi_2^n(a) = w \right\}. \tag{37}$$

3.4 Equivalence of Coalgebras and $\mathscr{C}$-Automata

Given the internal definitions of the semantic map from Equation (31) and the internal definition of the "accepted runs" semantics with Equation (37), we can now demonstrate that the semantics coincide.

Relation of $\mathscr{C}$-Automata to Coalgebras

It is necessary to clarify the relation of $\mathscr{C}$-automata to coalgebras, as this is not directly apparent. Specifically, we wish to demonstrate how to construct the "corresponding coalgebra" of a $\mathscr{C}$-automaton. This direction is preferable, as a coalgebra are more general, due to an absence of initial states.

Remark 3.11. For each $\mathscr{C}$-automaton $A = (Q, \Sigma, I, F, \delta)$, we can construct a coalgebra $\langle o, t \rangle : Q \longrightarrow \Omega \times Q^{\Sigma}$ by giving the definitions separately:

$$o(q) = (q \in F) \tag{38}$$

$$t(q) = \lambda\, s.\, \{\, q' \mid (q, s, q') \in \delta \,\} \tag{39}$$

Semantic Discrepancies

Before continuing, the two semantics have to be moulded to match each other, as the current presentation has two issues:

1. The "accepted runs" semantics partitions the accepted words by word length, while the semantic map denotes these in a single object,
2. the semantic map sends one state to a language, while the "accepted runs" semantic allows for automata with multiple initial states.

The first issue can either be resolved by aggregating the languages of each separate accepted run into a single object

$$\textstyle\coprod_{n<\omega} L_n(A) \xleftarrow{\iota_m} L_m(A) \tag{40}$$

or alternatively restricting the semantic map of the coalgebra to words of some fixed length:

$$\left\{\, w \colon \Sigma^n \;\middle|\; \bar{o}(\overline{t(q)}(w)) \,\right\} \subseteq \left\{\, w \colon \Sigma^* \;\middle|\; \bar{o}(\overline{t(q)}(w)) \,\right\}. \tag{41}$$

It should stand to reason, that either choice should allow for a straightforward proof of the other approach. For the sake of simplicity and saying in line with the finitist nature of toposes, we will follow the second approach.

As for the second problem, it is necessary to generalise the semantic map as to accept multiple initial states in the expected way (recall Equation (32)):

$$[\![I]\!]^*_n \coloneqq \{\, w \colon \Sigma^n \mid \exists\, q \in I.\, [\![q]\!]_n = w \,\} \tag{42}$$

Proof of Equivalence

Given the above adjustments, we are now in a position to clearly state the objective in the internal logic of $\mathscr{E}$. For some automaton $A = (Q, \Sigma, I, F, \delta)$, we intend to demonstrate that

$$[\![I]\!]^*_n = L^{(n)}(A) \tag{43}$$

holds, with n being the length of words.

Theorem 3.12. *The semantics of a $\mathscr{C}$-automaton and its corresponding trace semantics of a coalgebra coincide.*

Proof. We will derive the equivalent point-wise formulation of Equation (43) by case-distinction with regard to the length of a word w.

We begin by considering the empty word $w = \epsilon$: It is necessary to demonstrate that an initial state is also a final state:

$$
\begin{aligned}
& \epsilon \in [\![I]\!]^*_0 \\
\iff & \epsilon \in \left\{ w \colon \Sigma^0 \;\middle|\; \exists\, q \in I.\, \bar{o}(\overline{t(q)}(w)) \right\} \\
\iff & \epsilon \in \left\{ w \colon \Sigma^0 \;\middle|\; \exists\, q \in I.\, o(q) \right\} \\
\iff & \exists\, q \in I.\, o(q) \\
\iff & \exists\, q \in I.\, q \in F \\
\iff & \{ q \colon Q \mid q \in I \wedge q \in F \} \ncong \varnothing \\
\iff & I \cap F \ncong \varnothing \\
\iff & \epsilon \in \begin{cases} \{\} & \text{if } I \cap F \cong \varnothing \\ \{\epsilon\} & \text{otherwise} \end{cases} \\
\iff & \epsilon \in L^{(0)}(A)
\end{aligned}
$$

For a non-empty word $w = sw'$, the high-level reasoning looks like

$$
\begin{aligned}
& sw' \in [\![I]\!]^*_{n+1} \\
\iff & sw' \in \left\{ w \colon \Sigma^{n+1} \;\middle|\; \exists\, q \in I.\, \bar{o}(\overline{t(q)}(w)) \right\} \\
\iff & \exists\, q \in I.\, \bar{o}(\overline{t(q)}(sw')) \\
\iff & \exists\, q \in I.\, \bar{o}(\overline{t(t(q)(s))}(w')) \\
\overset{(*)}{\iff} & \exists\, a \in \mathsf{AccRun}^{(n+1)}_A.\, \pi_2^{n+1}(a) = sw' \\
\iff & sw' \in L^{(n+1)}(A)
\end{aligned}
$$

where the crucial step lies in $(*)$. We consider both directions separately:

The " $\implies$ " direction Given an initial state $q \in I$, such that processing sw' results in an final state, we have to construct an appropriate accepted run.

Knowing that from q we get to an accepting state q' in $n + 1$ steps by iterating over $sw' = s_1 s_2 \ldots s_n$, we can derive the intermediate states

$$
q = q_1 \xrightarrow{t(s_1)} q_2 \xrightarrow{t(s_2)} \ldots q_n \xrightarrow{t(s_n)} q_{n+1}
$$

This allows us to describe an $a \colon \mathsf{AccRun}^{(n)}_A$, by giving component of the accepted run

$$
\pi_i(a) = (q_i, s_n, q_{i+1}) \qquad \text{where } 1 \le i \le n.
$$

This satisfies the necessary conditions, as we know that by construction the witness $q_1 = q \in I$ ensures that $q_{n+1} = q' \in F$ holds.

The " $\impliedby$ " direction Now given an accepted run $a \colon \mathsf{AccRun}^{n+1}_A$, it is an intuitive choice to use $q = q_1 \coloneqq \pi_{1,1}(a)$ as the witness.

It is also clear that after traversing $sw' = s_1 s_2 \dots s_n = \pi_2^n(a)$, we will reconstruct the above chain of states,

$$
\begin{aligned}
q_2 &= \pi_{3,1}(a) \in t(q_1)(s_1) && \text{as } (q_1, s_1, q_2) = \pi_1(a) \in \delta \\
q_3 &= \pi_{3,2}(a) \in t(q_2)(s_2) \\
&\vdots \\
q_n &= \pi_{3,n-1}(a) \in t(q_{n-1})(s_{n-1}) \\
q_{n+1} &= \pi_{3,n}(a) \in t(q_n)(s_{n+1}) \\
q' =: q_{n+2} &= \pi_{3,n+1}(a) \in t(q_{n+1})(s_{n+1})
\end{aligned}
$$

where we know that $o(q')$ is true as by construction $q' = \pi_{3,n+1}(a) \in F$. ■

This concludes the internal proof that demonstrates that the accepted runs semantics coincides with the coalgebraic trace semantics, given that we can represent the language inside the topos.

4 $\mathscr{C}$-Automata and Graded Monads

The results in Chapter 3 require $\mathscr{E}$ to not only to be a topos, but also be countably extensive. This was necessary, as the Eilenberg-Moore semantics define a coalgebra map to an object $\Sigma^\star = \coprod_{i<\omega} \Sigma^i$, which is not a finite colimit and hence not constructable in an arbitrary topos (as for example **FinSets**).

In this chapter, we will present an alternative approach to defining the semantics of a coalgebra involving graded monads. This will involve defining a family of $n \in \mathbb{N}$ semantic maps, each determining the words of a language up to the depth n.

4.1 Graded Monads

We will be using the definition given by Milius, Pattinson and Schröder [MPS15]:

Definition 4.1. A *graded monad* on $\mathscr{C}$ is a family of endofunctors

$$(T_n \colon \mathscr{C} \longrightarrow \mathscr{C})_{n \in \mathbb{N}},$$

a natural transformation $\eta\colon \mathrm{Id} \Rightarrow T_0$ (*unit*) and a family of natural transformations (*multiplication*)

$$\left(\mu^{n,k} \colon T_n T_k \Rightarrow T_{n+k}\right)_{n \in \mathbb{N}, m \in \mathbb{N}}.$$

These satisfy the *unit* and *associativity laws*:

$$
\begin{array}{ccc}
T_n & \xrightarrow{\eta T_n} & T_0 T_n \\
{\scriptstyle T_n \eta}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu^{0,n}} \\
T_n T_0 & \xrightarrow[\mu^{n,0}]{} & T_n
\end{array}
\qquad\qquad
\begin{array}{ccc}
T_n T_k T_m & \xrightarrow{T_n \mu^{k,m}} & T_n T_{k+m} \\
{\scriptstyle \mu^{n,k} T_m}\downarrow & & \downarrow{\scriptstyle \mu^{n,k+m}} \\
T_{n+k} T_m & \xrightarrow[\mu^{n+k,m}]{} & T_{n+k+m}
\end{array}
$$

Milius, Pattinson and Schröder use graded monads as a means to encode "trace length" into the standard notion of coalgebraic trace semantics.

Definition 4.2. For a F-coalgebra (X, γ) the *graded trace semantics* consist of

- a graded monad $(T_n)_{n\in\mathbb{N}}$,
- a natural transformation $\alpha\colon G \Rightarrow T_1$.

Notation 4.3 ((Graded) Kleisli Star $(-)^*_n$)**.** For a $f\colon X \longrightarrow T_k Y$, we write

$$f^*_n = \mu^{n,k}_Y \circ T_n f\colon T_n X \longrightarrow T_{n+k} Y. \tag{44}$$

Definition 4.4. For a graded trace semantics $\big((T_n)_{n\in\mathbb{N}}, \alpha\big)$, the *$\alpha$-pretrace sequence* is a family of maps

$$\Big(\gamma^{(n)}\colon X \longrightarrow T_n 1\Big)_{n\in\mathbb{N}}$$

defined by

$$\begin{aligned} \gamma^{(0)} &\coloneqq \eta_X\colon X \longrightarrow T_0 1 \\ \gamma^{(n+1)} &\coloneqq (\gamma^{(n)})^*_1 \circ \alpha_X \circ \gamma\colon X \longrightarrow T_{n+1} 1 \end{aligned}$$

4.2 α-Pretrace Sequence in a Topos

The motivation behind using graded monads is that by restricting trace length, we can avoid depending on countably extensive toposes. Before giving a semantics via the α-pretrace sequence, we have to define an adequate graded monad in $\mathscr{E}$.

Here again, we will be reusing the idea from Milius, Pattinson and Schröder [MPS15, Ex. 5]: We take the graded monad in $\mathscr{E}$ to be

$$T \coloneqq \big(\mathbf{P}\left(\Sigma^{<n} + \Sigma^n \times -\right) : \mathscr{E} \longrightarrow \mathscr{E}\big)_{n\in\mathbb{N}}, \tag{45}$$

where for the sake of legibility we use the abbreviation

$$\Sigma^{<n} \coloneqq \coprod_{i=0}^{n} \Sigma^i.$$

The interpretation is that for each n we accumulate both accepted words of length strictly shorter than n (left injection) and the state of the coalgebra after having partially processed a word of length n (right injection).

To show that $(T_n)_{n\in\mathbb{N}}$ constitute a graded monad, we first have to define the unit and multiplication natural transformations and show that they are well-behaved. We interpret $\eta\colon \mathrm{Id} \Rightarrow T_0$ point-wise as a morphism mapping a state to a graded monad of depth 0. This means that the left injection under the power object does not yet contain any accepted words. Likewise, we know that after traversing the empty word, we would still remain in the same state. Hence, we can define unit internally as

$$\eta_X(q) \coloneqq \{\iota_2(\epsilon, q)\}. \tag{46}$$

As for multiplication of T_n and T_m, the main idea is that we want to extend the incomplete parses of T_n by at most m more steps. This means that the accepted words in T_n are preserved and extended by words which can be accepted within fewer than m more transitions. Words of length $n+m$ and their states become the new partial parses:

$$\begin{aligned}\mu_Q^{n,m}(S) \coloneqq\ & \{\, \iota_2(wv, V) \mid \iota_2(w, W) \in S, \iota_2(v, V) \in W \,\} \\ \cup\ & \{\, \iota_1(wv) \mid \iota_2(w, W) \in S, \iota_1(v) \in W \,\} \\ \cup\ & \{\, \iota_1(w) \mid \iota_1(w) \in S \,\},\end{aligned} \tag{47}$$

Given the structure, we now proceed to show that the monad satisfies the conditions given in Definition 4.1.

Proposition 4.5. *The above T_n satisfies the graded unit law.*

Proof. The statement of the unit law is that prepending or appending an arbitrary graded monad by T_0 has no computational effect. We show this by expanding and simplifying definitions until both prepending (ηM_n) and appending ($M_n \eta$) simplify back to the same expression. We begin by considering some $S\colon T_n Q$ and prepending ϵ,

$$\begin{aligned}& \mu_Q^{0,n}(\eta_Q(T_n)(S)) \\ =\ & \mu_Q^{0,n}(\{\iota_2(\epsilon, S)\}) \\ =\ & \{\, \iota_1(\epsilon w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(\epsilon w, q) \mid \iota_2(w, q) \in S \,\} \\ =\ & \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w, q) \mid \iota_2(w, q) \in S \,\} = S\end{aligned}$$

and then appending ϵ,

$$\begin{aligned}=\ & \mu_Q^{0,n}(T_n(\eta_Q)(S)) \\ =\ & \mu_Q^{0,n}(\{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w, \{\iota_2(\epsilon, q)\}) \mid \iota_2(w, q) \in S \,\}) \\ =\ & \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w\epsilon, q) \mid \iota_2(w, q) \in S \,\} \\ =\ & \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w, q) \mid \iota_2(w, q) \in S \,\} = S\end{aligned}$$

■

Proposition 4.6. *The above T_n satisfies the graded associativity law.*

Proof. The objective here is to show that the order of flattening more than two graded monads makes no difference. The following chain of equations is rather long-winded and verbose, but the intent is just apply Equation (47) to flatten all the words into either accepted words or partial parses.

As before, consider some $S\colon T_n T_k T_m Q$,

$$\begin{aligned}& \mu_X^{n,k+m}(T_n(\mu_X^{k,m})(S)) \\ =\ & \mu_X^{n,k+m}\left(\begin{aligned}&\{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ &\left\{\, \iota_2(w, x) \;\middle|\; \iota_2(w, S') \in S, x \in \mu_X^{k,m}(S') \,\right\}\end{aligned}\right) \\ =\ & \mu_X^{n,k+m}\left(\begin{aligned}&\{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ &\{\, \iota_2(w, \iota_1(w')) \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ &\{\, \iota_2(w, \iota_1(w'w'')) \mid \iota_2(w, S') \in S, \iota_2(w', q) \in S', \iota_1(w'') \in S'' \,\} \cup \\ &\{\, \iota_2(w, \iota_2(w'w'', q)) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_2(w'', q) \in S'' \,\}\end{aligned}\right)\end{aligned}$$

$$= \mu_X^{n,k+m} \begin{pmatrix} \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ \{\, \iota_2(w, \iota_1(w')) \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ \{\, \iota_2(w, \iota_1(w'w'')) \mid \iota_2(w, S') \in S, \iota_2(w', q) \in S', \iota_1(w'') \in S'' \,\} \cup \\ \{\, \iota_2(w, \iota_2(w'w'', q)) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_2(w'', q) \in S'' \,\} \end{pmatrix}$$

$$= \begin{pmatrix} \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ \{\, \iota_1(ww') \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ \{\, \iota_1(ww'w'') \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_1(w'') \in S'' \,\} \cup \\ \{\, \iota_2(ww'w'', q) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_2(w'', q) \in S'' \,\} \end{pmatrix}$$

$$= \mu_X^{n+k,m} \begin{pmatrix} \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ \{\, \iota_1(ww') \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ \{\, \iota_2(ww', \iota_1(w'')) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_1(w'') \in S'' \,\} \cup \\ \{\, \iota_2(ww', \iota_2(w'', q)) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_2(w'', q) \in S'' \,\} \end{pmatrix}$$

$$= \mu_X^{n+k,m} \begin{pmatrix} \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ \{\, \iota_1(ww') \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ \{\, \iota_2(ww', \iota_1(w'')) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_1(w'') \in S'' \,\} \cup \\ \{\, \iota_2(ww', \iota_2(w'', q)) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', \iota_2(w'', q) \in S'' \,\} \end{pmatrix}$$

$$= \mu_X^{n+k,m} \begin{pmatrix} \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \\ \{\, \iota_1(ww') \mid \iota_2(w, S') \in S, \iota_1(w') \in S' \,\} \cup \\ \{\, \iota_2(ww', x) \mid \iota_2(w, S') \in S, \iota_2(w', S'') \in S', x \in S'' \,\} \end{pmatrix}$$

$$= \mu_X^{n+k,m}(\mu_X^{n,k}(T_m)(S)). \qquad \blacksquare$$

Having established that T_n is a graded monad, the next step towards the graded semantics is to give a natural transformation $\alpha \colon G \Rightarrow T_1$. We define α internally and component-wise:

$$\alpha(\langle o, t \rangle) = \{\, \iota_2(s, q) \mid t(s) = q \,\} \cup \begin{cases} \{\iota_1\, \epsilon\} & \text{if } o = \text{true} \\ \{\} & \text{otherwise} \end{cases} \tag{48}$$

Proposition 4.7. *α is natural.*

Proof. This is easy to see as

$$T_1 = \mathbf{P}\left(\Sigma^{<1} + \Sigma^1 \times -\right) \cong \mathbf{P}\left(1 + \Sigma \times -\right) \cong \mathbf{P}\left(1\right) \times \mathbf{P}\left(\Sigma \times -\right) \cong \Omega^1 \times \left(\mathbf{P}\, -\right)^{\Sigma} = G$$

indicates that α is a natural isomorphism, and hence also a natural transformation. $\blacksquare$

Thus we have verified all the prerequisites have to be met to construct the α-pretrace sequence using the graded trace semantics. We can instantiate Definition 4.4 with $T_n Q$ and the coalgebra $\gamma = \langle o, t \rangle = \Omega \times (\mathbf{P}(-))^{\Sigma}$ as

$$\begin{aligned} \gamma^{(0)}(q) &= \eta_Q(q) = \{\iota_2(\epsilon, q)\} \\ \gamma^{(n+1)}(q) &= \left(\left(\gamma^{(n)}\right)_1^* \circ \alpha_{GQ} \circ \gamma\right)(q) \\ &= \{\, \iota_1(\epsilon) \mid (o)(q) \,\} \\ &\quad \cup \left\{\, \iota_1(sw) \;\middle|\; \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(n)}(q') \,\right\} \\ &\quad \cup \left\{\, \iota_2(sw, q'') \;\middle|\; \exists\, q' \in t(q)(s).\, \iota_2(w, q'') \in \gamma^{(n)}(q') \,\right\}. \end{aligned} \tag{49}$$

4.3 α-Pretrace Sequence and Accepted Run Semantics

We can consider two approaches to demonstrate that the graded semantics via the α-pretrace sequence and the accepted runs semantics from Section 3.3 coincide:

1. As the graded semantics accumulate all accepted words with length less than some n, while the accepted runs semantics partitions the language by word length, one could accumulate all the accepted runs up to n and compare the two subobjects

$$\coprod_{i=0}^{n} L^{(i)}(A) = \left\{ w\colon \Sigma^{<n+1} \;\middle|\; \exists\, q \in I.\, \iota_2(w) \in \gamma^{(n+1)}(q) \right\},$$

2. Instead we could consider restricting the graded semantics, and only considering words of some length $m \leq n$. These can then compared to the accepted word semantics of some fixed length m:

$$L^{(m)}(A) = \left\{ w\colon \Sigma^{m} \;\middle|\; \exists\, q \in I.\, \iota_2(w) \in \gamma^{(n+1)}(q) \right\} \qquad \text{for all } m \leq n.$$

We shall take the second path, and afterwards use it to derive the first option.

Notation 4.8. We will give the subobject comprehension used above a name:

$$[\![I]\!]^{\gamma}_{n} = \left\{ w\colon \Sigma^{<n} \;\middle|\; \exists\, q \in I.\, \iota_2(w) \in \gamma^{(n+1)}(q) \right\} \tag{50}$$

Theorem 4.9. *For some depth n, the accepted words of length $m \leq n$ in the α-pretrace sequence coincide with the accepted runs of a $\mathscr{C}$-automaton.*

Proof. Recall that as pointed out in Remark 3.11, that each $\mathscr{C}$-automaton $A = (Q, \Sigma, I, F, \delta)$ has a corresponding coalgebra $\gamma = \langle o, t\rangle : Q \longrightarrow \Omega \times \mathbf{P}(Q)^{\Sigma}$.

We first have to distinguish between empty and nonempty words. For empty words, where $m = 0$, we know that the accepted runs semantics depends on the intersection of the initial and final states to be non-empty if ϵ is to be accepted. As for the α-pretrace sequence, we know $\gamma^{(1)}(q)$ contains $\iota_1(\epsilon)$ if $q \in F$, and that in Equation (50) q is an element of I:

$$\begin{aligned}
&\epsilon \in L^{(0)}(A) \\
\iff& I \cap F \ncong \varnothing \\
\iff& \exists\, q \in I.\, q \in F \\
\iff& \exists\, q \in I.\, o(q) \\
\iff& \exists\, q \in I.\, \iota_1(\epsilon) \in \{\, \epsilon \mid o(q) \,\} \\
\iff& \exists\, q \in I.\, \iota_1(\epsilon) \in (\gamma^{(1)})(q) \\
\iff& \epsilon \in \left\{ w\colon \Sigma^{0} \;\middle|\; \exists\, q \in I.\, \iota_1(w) \in \gamma^{(1)}(q) \right\} \\
\iff& \epsilon \in [\![I]\!]^{\gamma}_{0} \cap \Sigma^{0}
\end{aligned}$$

As for nonempty word $w = s_1 \ldots s_m$, we want to argue that whenever there is an accepting run $a \in \mathsf{AccRun}^{m}_{A}$, that underneath the α-pretrace sequence there must exist a chain of transitions via t using w from a initial state to an accepting state:

$$
\begin{aligned}
& w \in L^{(m)}(A) \\
\iff & \exists\, a \in \mathsf{AccRun}_A^{(m)}.\, \pi_2^m(a) = w \\
\overset{(*)}{\iff} & \exists\, q \in I.\, \iota_1(w) \in \gamma^{(m+1)}(q) \\
\iff & w \in \left\{ w \colon \Sigma^m \,\middle|\, \exists\, q \in I.\, \iota_1(w) \in \gamma^{(m+1)}(q) \right\} \\
\iff & w \in [\![I]\!]_m^\gamma \cap \Sigma^m
\end{aligned}
$$

We consider the $(*)$-step in both directions separately:

The " $\implies$ " direction Given an accepting run, we wish to construct a chain of transitions. First, we have to give a witness for the existential claim: We use $q \coloneqq \pi_{1,1}(a)$, as we know by construction that the this state must be initial.

To show $\iota_1(w) \in \gamma^{(m)}(q')$, we recall that this means there is a chain of transitions

$$
q =: q_1 \xrightarrow{t(s_1)} q_2 \xrightarrow{t(s_2)} \dots \xrightarrow{t(s_{m-1})} q_m \xrightarrow{t(s_m)} q_{m+1}
$$

by the underlying transition morphism t, where $q_{m+1} \in F$ is an accepting state.

Indeed, this is the case as the accepting run correspond to the chain of transitions in the expected way

$$
q =: \pi_{1,1}(a) \xrightarrow{\pi_{2,1}(a)} \pi_{1,2}(a) \xrightarrow{\pi_{2,2}(a)} \dots \xrightarrow{\pi_{2,m-1}(a)} \pi_{1,m}(a) \xrightarrow{\pi_{2,m}(a)} \pi_{3,m}(a),
$$

where $q_{n+1} \coloneqq \pi_{3,n}(a)$ is an accepting state by construction.

The " $\impliedby$ " direction Given a chain of transitions as above, we can construct a well-formed accepting run a defined element-wise as

$$
\pi_i(a) = (q_i, w_i, q_{i+1}), \qquad \text{for all } 1 \leq 0 \leq m
$$

where it is easy to see that $\pi_2^m(a) = w$ holds. ■

Before proceeding to show that the accepted runs semantics aggregates to the graded semantics, we will need to assure ourselves that the graded semantics is "well-behaved" in the sense that increasing depths only extend the border of the language. We want to use this property, as the equality of all words will result from the equality of all words of equal length, which we can derive from Theorem 4.9. We will prove this by first showing that a greater element in the α-pretrace sequence does not loose any words, and then that it only adds longer word.

Lemma 4.10. *Increasing depths of the α-pretrace semantics preserve all prior words,*

$$
[\![I]\!]_n^\gamma \subseteq [\![I]\!]_{n+1}^\gamma\,,
$$

for any subobject $I \subseteq Q$ and any $n \in \mathbb{N}$.

Proof. Intuitively, we understand that increasing the depth of a graded monad, any accepting state reachable over $w = s_1 \dots s_n$ is still accessible if we allow ourselves to take more steps. Note that this is not the case for $\exists\, q \in I.\, \gamma^{(n)}(q) \subseteq \gamma^{(n+1)}(q)$, as the partial parses are lost when increasing the depth of the graded monad.

To prove this, we proceed by induction over n, beginning with $n = 0$:

$$
\begin{aligned}
&[\![I]\!]^{\gamma}_{0} = \left\{ w \;\middle|\; \exists\, q \in I.\, \iota_1(w) \in \gamma^{(1)}(q) \right\} \\
&= \left\{ \epsilon \;\middle|\; \exists\, q \in I.\, \bar{o}(\overline{t(q)}(\epsilon)) \right\} \\
&= \{ \epsilon \mid \exists\, q \in I.\, o(q) \} \\
&\subseteq \{ \epsilon \mid \exists\, q \in I.\, o(q) \} \cap \left\{ sw \;\middle|\; \exists\, q \in I.\, \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(0)}(q') \right\} \\
&= \left\{ w \;\middle|\; \exists\, q \in I.\, \iota_1(w) \in \gamma^{(2)}(q) \right\} = [\![I]\!]^{\gamma}_{1}
\end{aligned}
$$

and then proceed with the inductive step:

$$
\begin{aligned}
&[\![I]\!]^{\gamma}_{n} = \left\{ w \;\middle|\; \exists\, q \in I.\, \iota_1(w) \in \gamma^{(n+1)}(q) \right\} \\
&= \{ \epsilon \mid \exists\, q \in I.\, o(q) \} \cup \left\{ sw \;\middle|\; \exists\, q \in I.\, \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(n)}(q) \right\} \\
&\subseteq \{ \epsilon \mid \exists\, q \in I.\, o(q) \} \cup \left\{ sw \;\middle|\; \exists\, q \in I.\, \exists\, q' \in t(q)(s).\, \iota_1(w) \in \gamma^{(n+1)}(q) \right\} \qquad \text{by I.H.} \\
&= \left\{ w \;\middle|\; \exists\, q \in I.\, \iota_1(w) \in \gamma^{(n+2)}(q) \right\} = [\![I]\!]^{\gamma}_{n+1}
\end{aligned}
$$

Note that the induction hypothesis is still applicable, as we are not using $\exists\, q \in I.\, \gamma^{(n-1)}(q) \subseteq \gamma^{(n)}(q)$ in general, but only on the left injections. ■

Lemma 4.11. *Each level $n \geq 1$ of the α-pretrace semantics only adds words of length n:*

$$
[\![I]\!]^{\gamma}_{n+1} \setminus [\![I]\!]^{\gamma}_{n} \subseteq \Sigma^{n+1},
$$

for any subobject $I \subseteq Q$ and $n \in \mathbb{N}$.

Proof. For any particular n and $w = s_1 \dots s_m$ (where $m \leq n$), we can unfold $w \in [\![I]\!]^{\gamma}_{n}$ to a proposition of the form

$$
\exists\, q \in I.\, \exists\, q_1 \in t(q)(s_1).\, \exists\, q_2 \in t(q)(s_2).\, \dots\, \exists\, q_{m+1} \in t(q_m)(s_m).\, o(q_{m+1}),
$$

where the number of existential quantifiers is bound by n.

For $w = s_1 \dots s_m \in [\![I]\!]^{\gamma}_{n+1} \setminus [\![I]\!]^{\gamma}_{n}$ to hold, it is obvious that for any $m \leq n$ if $w \in [\![I]\!]^{\gamma}_{n+1}$ we have a chain of states $q, q_1, q_2, \dots, q_{m+1}$ which one could also pick for $w \in [\![I]\!]^{\gamma}_{n}$ to hold. Only words of length exactly $n+1$ would not be affected, as there is no chain of states that could be constructed with strictly less than $n+1$ witnesses. We can therefore conclude, that all words in the difference must be of length $n+1$. ■

Having established these properties of the α-pretrace sequence, we have the means to state and easily prove the first equivalence:

Corollary 4.12. *The accepted runs semantics up to n aggregates to the same words as the α-pretrace sequence:*

$$
\coprod_{i=0}^{n} L^{(i)}(A) \cong [\![I]\!]^{\gamma}_{n}
$$

Proof. By Lemmas 4.10 and 4.11 we can partition $[\![I]\!]_n^\gamma$ by the length of each word, and compare these separately

$$L^{(i)}(A) = [\![I]\!]_n^\gamma \cap \Sigma^i \qquad \text{for } i \leq n,$$

which holds by Theorem 4.9. ■

This concludes the analysis of graded semantics, and gives us an alternative perspective on the semantics of nondeterministic automata in an arbitrary topos $\mathscr{E}$ without having to assume that $\mathscr{E}$ is countably extensive.

5 Summary

We have presented two different approaches to equate the semantics of a $\mathscr{C}$-automaton with coalgebraic trace semantics.

The first approach (Section 2.3) restricted our assumptions by assuming that the topos $\mathscr{E}$ is countably extensive, i.e. with countable coproducts, which was necessary to speak of a language Σ^* with words of arbitrary length. The remaining definitions and arguments were discussed inside the internal logic of $\mathscr{E}$. The result was conclusive in that given the additional structure, the two semantics coincide (Theorem 3.12).

Yet as this result relies on the additional assumption, it is of interest to take a different approach as well: Section 4.2 utilised graded monads to describe the α-pretrace semantics of a coalgebra. In our case, this give us a family of "depth-limited" semantic maps, each of which gives a finite language. By shifting the countable infinity outside of the internal logic of a topos, we were able to generalise the results to an arbitrary topos. This means that the results are also applicable in a non-countable extensible topos, whereas the first approach was not. Finally, we have shown that the α-pretrace semantics relate to the accepted runs semantics in a sensible way (Theorem 4.9 and Corollary 4.12).

An interesting followup question is to what degree these results can be replicated in a category with even less structure. Examples includes pretoposes or Heyting categories [Joh02, A.1.4, p. 39] (or "Logos" as referred to by Freyd and Scedrov [FS90, Sec. 1.7, p. 117]). For the latter, the internal language is that of first order logic (as opposed to higher order logic in a topos). Amongst other things, one would have to show that the language morphism $\mathsf{AccRun}_A^{(n)} \longrightarrow \Sigma^n$ has an (epi, mono)-factorisation, as this is not a property of all morphisms have in a Heyting category.

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Acknowledgements

I would like to thank Stefan Milius and Florian Frank for their regular assistance in writing this thesis. Their patience when answering my questions assisted not only my understanding of the topic at hand, but also to deepen my understanding of category theory as such and the mathematical mindset in general.

My gratitude also extends to my family and friends, who took the time to proofread my drafts; in no particular order: Katharina H., Florian G., Eva W., Boris K., Tara L., Uwe Lukas H., Max Ole E., Lorena K., Ksenia K.