An Excordium on Computational Trinitarianism[1]

Curry-Howard-Lambek Correspondence

As revealed by KALUĐERČIĆ, Philip;

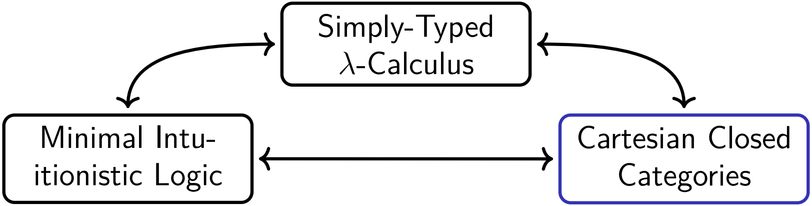
Questions or Complaints? Mail `philip.kaludercic` *at* `fau.de`.

2024-12-19, last typeset December 27, 2024, 22:01

1 Available on the WWW: `https://wwwcip.cs.fau.de/~oj14ozun/src+etc/chl.pdf`

Abstract

Goal: We want to extend the *Logic-Language* correspondance by *Categories*:



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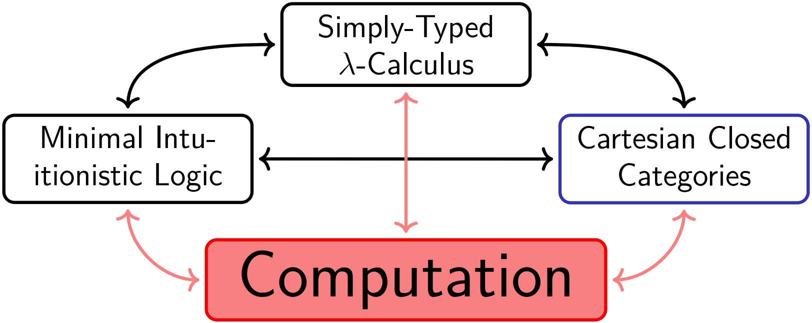


Each edge represents a “moment” of the essence of computation?

Part I

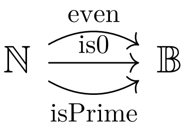
Yet Another Introduction to Category Theory

Section 1

The Definition of a Category

Set-theoretic functions | Posets | Monoids

Functions “connect” Sets



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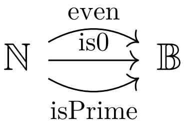

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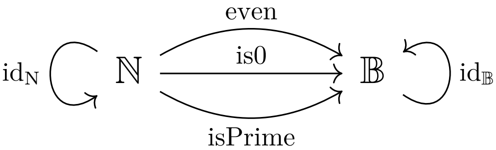

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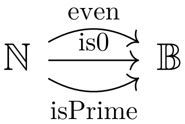

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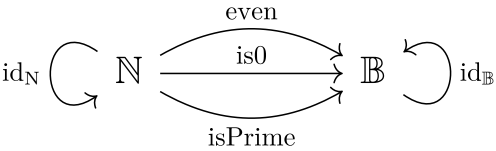
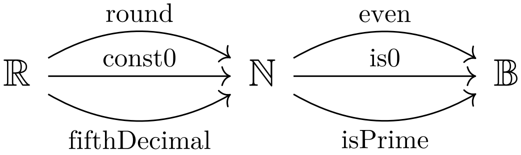


Functions can be composed (associativley)



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A relation “connect” elements

$$\{a, b\} \subseteq \{a, b, c\}$$

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The relation is transitive

$$\{a\} \subseteq \{a, b\} \subseteq \{a, b, c\} \quad \text{and}$$

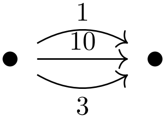
$$\{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\}$$

grants

$$\{a\} \subseteq \{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\}$$

Set-theoretic functions | Posets | Monoids

Elements of a monoid (e.g. $(\mathbb{N}, +, 0)$) "connect"



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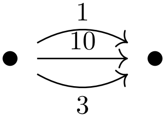
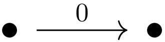


There is a unique neutral "arrow"



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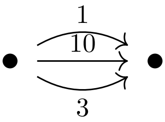

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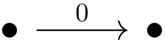
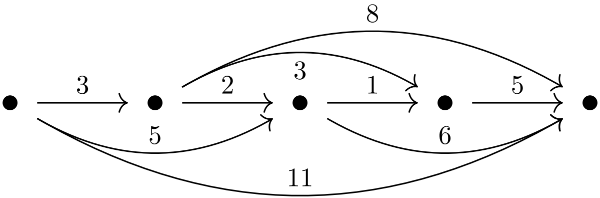


All "arrows" are associative



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$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A} \text{ (neutral)}$$

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s.t. $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

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This is a canonical example of a category. Many other examples restrict $\mathbf{Sets}$ to specific objects and functions ($\mathbf{FinSet}$, $\mathbf{Top}$, $\mathbf{Gra}$, $\mathbf{Grp}$) or generalise it ($\mathbf{Rel}$, $\mathbf{Par}$).

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This example illustrates that arrows are not always function-*ish*.

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This example emphasises the "monoidal" nature of categories.

Section 2

Selected Universal Properties of Constructions

Fact (Fun)

Category theory allows us to recognise different settings where objects relate (via arrows) to one another in analogous ways.

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Fact (Fun, *continued*)

Of particular interest are constructions that are identified by a unique arrow.

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Definition

A category $\mathscr{C}$ with a terminal object $1 \in \mathrm{Ob}\,(\mathscr{C})$ has exactly one arrow

$$!\colon A \longrightarrow T, \qquad |\mathrm{Hom}_{\mathscr{C}}\,(A, T)| \cong \{*\}$$

for every other object $A \in \mathrm{Ob}\,(\mathscr{C})$.

Category **Sets**

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A product $A \times B$ of two sets has two projections

$$\begin{aligned} \pi_1 \colon A \times B &\to A \\ (a, b) &\mapsto a \\ \pi_2 \colon A \times B &\to B \\ (a, b) &\mapsto b \end{aligned}$$

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Definition (Preliminary?)

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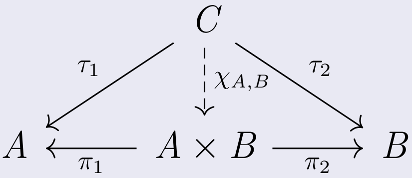

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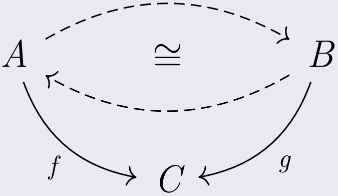
Both are equally well fit and are mutually correspond to one another.

Fact (...up to isomorphism)

When thinking categorically and considering the relations of objects over the contents of the objects, we handle objects within a equivalence class of "isomorphisms".



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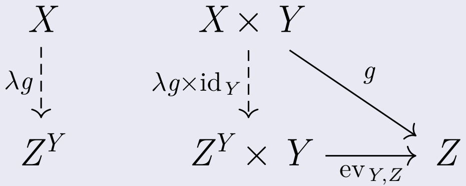

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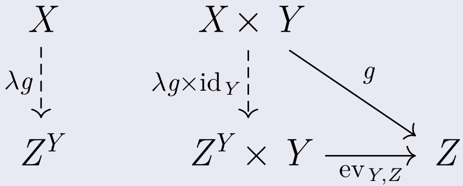

Example

In $\mathbf{Sets}$ B^A is represents all functions from A to B.

The aforementioned can be expressed as the equation:

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Do you recognise this from somewhere?

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The exponential object in Heyting Algebra following from the above, corresponds to the well-known definition of implication:

$$a \sqcap b \sqsubseteq c \iff a \sqsubseteq b^c$$

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Categories that satisfy these properties include $\mathbf{Sets}$, categories of Heyting Algebras.

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Now what does all of this have to do with the λ-Calculus or (positive/minimal) intuitionist logic?

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- Twisted Generalized Cohomology in Linear Homotopy Type Theory, ...

Part II

Equational Theories and λ-Theories

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$$\frac{A = B \qquad B = C}{A = C}\ (\mathsf{trans})$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

We write

$$\Gamma \vdash s = t \colon A$$

to state that $s \colon A$ and $t \colon A$ are equal in the same context Γ.

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...rules for product and unit types ...

Part III

Qu'est-ce qui correspond à quoi ?

(What does correspond to what?)

Fact

The general idea of the correspondence is...

$$\textit{Types} \iff \textit{Objects}$$

$$\textit{Terms} \iff \textit{Arrows}$$

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So demonstrating the existence of a arrow is the same "moral act" as a constructive proof of a proposition or inhabiting a type.

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4. *For any arrow $g\colon A \times B \longrightarrow C$ we have a corresponding $\lambda g\colon A \longrightarrow C^B$ (and vice versa).*
5. *For any A and B^A we have a $\mathrm{ev}_{A,B}\colon A \times B^A \longrightarrow B$*

λ-Calculus CCC

λ-Calculus

Application of terms:

$$s\colon A \to B, t\colon A \vdash st\colon B$$

CCC

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$\vdash \lambda\, x.\, t\colon A \to B$	$\lambda g\colon 1 \longrightarrow B^A$
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With the obvious correspondences between product types and categorical products ($\texttt{fst} \approx \pi_1$, $\texttt{snd} \approx \pi_2$, ...).

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by the term

$$\lambda\, p.\, \lambda\, a.\, ((\texttt{fst}\ p)a, (\texttt{snd}\ p)(\texttt{fst}\ p))$$

Proof.

Obvious, duh. □

We can prove the intuitionistic proposition is satisfiable

$$(A \to B) \land ((A \to B) \to C) \to A \to B \land C$$

Proof.

... by constructing the proof tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{(A \to B) \land ((A \to B) \to C)}}{A \to B}(\land_{E1}) \quad \overline{A}}{B} \qquad \dfrac{\dfrac{\overline{(A \to B) \land ((A \to B) \to C)}}{(A \to B) \to C}(\land_{E2}) \quad \dfrac{\overline{(A \to B) \land ((A \to B) \to C)}}{A \to B}(\land_{E1})}{C}}{B \land C}}{A \to B \land C}(\to_I)}{(A \to B) \land ((A \to B) \to C) \to A \to B \land C}(\to_I)$$

□

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that in turn is given by

$$f := \left\langle \mathrm{ev}_{A,B} \circ \langle \pi_2, \pi_1 \rangle , \mathrm{ev}_{B^A,C} \circ \langle \pi_3, \pi_2 \rangle \right\rangle \qquad \square$$

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \rightarrow B) \rightarrow (B \rightarrow A)$.

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Proof "by exhaustion".

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$$A^{B} \xrightarrow{\lambda f} B^{A}$$

$$A \times A^{B} \xrightarrow{\lambda\lambda f} B$$

doesn't leave us with any means to construct arrow in an arbitrary CCC (recall the "Intermission"). □

Essentially we are giving a categorical interpretation of λ terms:

$$
\begin{aligned}
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} \\
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To ensure that the interpretation is sound and complete we need to prove that the rules of the λ theory $\mathbb{T}$ coincide with arrow-equality.

Definition

We say that λ-Calculus is the internal language of Cartesian Closed Categories.

Part IV

Pour aller plus loin

(To go further)

Fact

The more "structure" a category has, the more interesting the internal logic[2]*:*

[2] See https://ncatlab.org/nlab/show/internal+logic

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- *A* *topos* *(more on that in a moment) models to* *finitist, intuitionistic higher-order logic*

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- *A boolean topos (ie. with well-behaved complements) models to classical higher-order logic*
- *A symmetric monoidal category (generalisation of CCC) models to linear logic*

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Further Reading and Sources I

Recommended Reading on Category Theory

- https://arxiv.org/pdf/1612.09375
- Book “Categories for the working mathematician” (Mac Lane)
- Book “ Basic Category Theory for Computer Scientists” (Pierce)
- https://web.archive.org/web/20230301160845/https://people.math.harvard.edu/~mazur/preprints/when_is_one.pdf

Recommended Reading on Categorical Logic

- https://awodey.github.io/catlog/notes/ (WIP)

Further Reading and Sources II

- https://arxiv.org/abs/1102.1313
- https://plato.stanford.edu/entries/lambda-calculus/#LThe
- https://golem.ph.utexas.edu/category/2006/08/cartesian_closed_categories_an_1.html
- Book "Introduction to Higher Order Categorical Logic" (Lambek)
- Book "The Lambda Calculus, its Syntax and Semantics" (Barendregt)
- Book "Topoi: The Categorial Analysis of Logic" (Goldblatt)

Further Reading and Sources III

Related and more complicated concepts

- https://math.ucr.edu/home/baez/rosetta.pdf#page=66
- Book "Elementary Categories, Elementary Toposes" (McLarty)
- Book "Sketches of an Elephant" (Johnstone)
- Book "Sheaves and Geometry in Logic" (Mac Lane)
- Book "Handbook of Categorical Algebra" (Borceux), specifically Volume 3

A possible first step in the research program is 1700 doctoral theses called "A Correspondence between x and Church's -notation".

— A popular joke