$\mathcal{EM}$-Style Semantics in a Topos $\mathscr{E}$

Philip Kaluđerčić
philip.kaludercic@fau.de

17Jun24, typeset on July 26, 2024

We follow Jacobs, et. al.,[1] transliterating the proof from $\mathbf{Sets}$ into an arbitrary Topos $\mathscr{E}$, specifically trying to express the map $\wp(X) \longrightarrow \wp(A^\star)$ in terms of the internal logic of $\mathscr{E}$.

Review of $\mathcal{EM}$-Style Non-Deterministic Automata In $\mathbf{Sets}$, we can model a non-deterministic automaton as the morphism

$$X \longrightarrow 2 \times \wp(X)^\Sigma,$$

where we can express $2 \times \wp(X)^\Sigma$ as the composition of the functor $2 \times -^\Sigma$ and the (powerset) monad $\wp(-)$.

An Eilenberg-Moore category $\mathcal{EM}(T)$ of a monad (T, η_X, μ_X) of a category $\mathscr{C}$, has

1. as objects, morphisms in $\mathscr{C}$ of the form $a\colon T(X) \longrightarrow X$, such that $a \circ \eta_X = \mathrm{id}_X$ and $a \circ T(a) = a \circ \mu_X$ hold,
2. as morphisms between objects $x\colon T(X) \longrightarrow X$ and $y\colon T(Y) \longrightarrow Y$, morphism $f\colon X \longrightarrow Y$ from $\mathscr{C}$ such that $b \circ T(f) = f \circ a$.

In other words, we are considering a sub-category of F-Algebra, for a monad T with the above placing conditions on objects.

For a category $\mathscr{C}$, assume the following in order:

- An arbitrary endofunctor $G\colon \mathscr{C} \longrightarrow \mathscr{C}$,
- An arbitrary monad $(T\colon \mathscr{C} \longrightarrow \mathscr{C}, \eta, \mu)$,
- an $\mathcal{EM}$-law $\rho : TG \Rightarrow GT$,
- and by the corresponding lifting a endofunctor

  $$\hat{G}\colon \mathcal{EM}(T) \longrightarrow \mathcal{EM}(T),$$

- a final G-coalgebra $\zeta \in \mathrm{Hom}_{\mathscr{C}}(Z, GZ)$
- a G-coalgebra $\rho \circ T(\zeta) \in \mathrm{Hom}_{\mathscr{C}}(TZ, GTZ)$,
- a unique map $\alpha\colon TZ \longrightarrow Z$ in from $\rho \circ T(\zeta)$ to ζ, due to finality of ζ,

Then ζ may as well be a final coalgebra in $\mathcal{EM}(T)$, of the form

$$\alpha \mapsto \hat{G}(\alpha) : (TZ \longrightarrow Z) \longrightarrow (TZ \longrightarrow GTZ),$$

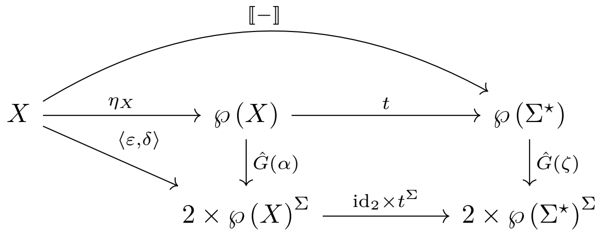
where $\hat{G}(\alpha) = \rho_X(G(\alpha))$.

For a non-deterministic automaton described by $G\colon X \longrightarrow 2 \times \wp(X)^\Sigma$, where the final coalgebra is $Z = \wp(\Sigma^\star)$ (set of accepted words) is also final for $\hat{G}\colon \wp(X) \longrightarrow 2 \times \wp(X)^2$. For a given state X we can determine the set of accepted words by composing the monadic unit $\eta_X\colon X \longrightarrow \wp(X)$, i.e. $\eta_X(x) = \{\, y \mid y = x \,\} = \{x\}$ with $\hat{G}$, resulting in the semantic map

$$[\![-]\!] : X \longrightarrow \Sigma^\star$$

in the base category, defined by

[1] Bart Jacobs, Alexandra Silva, and Ana Sokolova. "Trace semantics via determinization". In: *International Workshop on Coalgebraic Methods in Computer Science*. Springer. 2012, pp. 109–129.



Translation into an arbitrary Topos $\mathscr{E}$ We want to generalise $[\![-]\!]$ from $\mathbf{Sets}$ into $\mathscr{E}$. Knowing that in the internal logic

$$\eta_X(x) = \{\, y \mid y = x \,\},$$

the main issue remains to express $t\colon \mathbf{P}X \longrightarrow \mathbf{P}\Sigma^\star$. To this end, we first have to determine the nature of $\Sigma^\star$.

Going by Frank, et. al.,[2] we could intuitively define

$$\Sigma^\star \coloneqq \coprod_{n\in\mathbb{N}} \Sigma^n = \coprod_{n\in\mathbb{N}} \underbrace{\Sigma \times \cdots \times \Sigma}_{n \text{ times}},$$

but that requires $\mathscr{E}$ to be "countably extensive" (supporting countable coproducts), which is not grated in general, considering that general toposes allow for finite cocompleteness.

Instead, Frank, et. al. define a *language* as a family of subobjects

$$L \coloneqq (m_n^{(L)}\colon L^{(n)} \rightarrowtail \Sigma^n)_{n\in\mathbb{N}},$$

where $L^{(n)}$ denotes the words of length n, and $L \leq L'$ is defined point-wise.

Here the question arises, of how we can express "$\mathbf{P}L$"? The notion of a family over $\mathbb{N}$, which is countably infinite, cannot be articulated in an arbitrary, non-countably extensive topos, as the family of subobjects would correspond directly to a countable coproduct.

So should we instead consider $[\![-]\!]_n : X \longrightarrow \mathbf{P}\left(\Sigma^n\right)$, that describes accepted words of length n from a given state X? This would result in a semantic given by a family of $[\![-]\!]_n$ maps.

Recall that in general $\mathbf{P}A \cong 1 \rightarrowtail \mathbf{P}A$ correspond[3] to subobjects $m\colon S \rightarrowtail A$. In our case, this means we are trying to find

$$s_n^{(L)}\colon 1 \rightarrowtail \mathbf{P}\left(\Sigma^n\right) \qquad \rightsquigarrow \qquad m_n^{(L)}\colon L^{(n)} \rightarrowtail \Sigma^n.$$

By using a map reminiscent of the usual map from a coalgebra of a non-deterministic automaton to the terminal coalgebra (indicated by t in the above diagram), we can directly describe the subobject of accepted words in Σ^n of a state $x \in X$ in the internal logic of $\mathscr{E}$:

$$[\![x]\!]_n = \{\, (\sigma_1, \ldots, \sigma_n) \mid \varepsilon(\delta^n(\eta_X(x))(\sigma_1, \ldots, \sigma_n)) \,\}\}$$

which matches the intended type above, where[4]

$$\delta^n(S) = (\sigma_1, \ldots, \sigma_n) \mapsto \delta^{n-1}(\mu_X(\{\, \delta(x)(\sigma_1) \mid x \in S \,\}))(\sigma_2, \ldots, \sigma_n)$$

for $n > 0$, and otherwise

$$\delta^0(S) = S.$$

As we have a $x \in X$ given, we can also describe it using global element $x\colon 1 \rightarrowtail X$. By composing this with $[\![-]\!]_n$, we have a description of

$$s_n^{(n)} = [\![-]\!]_n \circ x, \qquad \text{read "}[\![x]\!]_n\text{"}.$$

How does this stand in relation to $m_n^{(L)}$? Fundamentally, this relies on the above quoted observation

$$\mathrm{Sub}_{\mathscr{E}}(A) \underset{(\aleph)}{\cong} \mathrm{Hom}_{\mathscr{E}}(A, \Omega) \underset{(\beth)}{\cong} \mathrm{Hom}_{\mathscr{E}}(1, \mathbf{P}A),$$

[2] Florian Frank, Stefan Milius, and Henning Urbat. *Positive Data Languages*. 2023. arXiv: 2304.12947 [cs.FL], p. 10.

[3] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012, p. 165.

[4] Note that in this case ε and δ do not have the domain X, but $\mathbf{P}X$, and hence can be defined as $\varepsilon = \pi_1 \circ \hat{G}(\alpha)$ and $\delta = \pi_2 \circ \hat{G}(\alpha)$, for the coalgebra α representing the automaton, and $\hat{G}$ lifts from $\mathscr{E}$ to $\mathcal{EM}(\mathbf{P}\,(-))$.

which is natural in A, or specifically in our case for $n \in \mathbb{N}$

$$\mathrm{Sub}_{\mathscr{E}}(\Sigma^n) \cong \mathrm{Hom}_{\mathscr{E}}(\Sigma^n, \Omega) \cong \mathrm{Hom}_{\mathscr{E}}(1, \mathbf{P}\Sigma^n).$$

The ($\beth$) correspondence is just exponential transposition, that is easily seen when one remembers that $\mathbf{P}A = \Omega^A$. To understand ($\aleph$), one has to recall that $\mathrm{Sub}_{\mathscr{E}}(A)$ is the lattice of subobjects of A. As $\mathrm{Sub}_{\mathscr{E}}(A)$, like all categories of a poset are thin categories, there is at most one morphism between two objects, where each morphism is a mono (and epi). While usually we have a unique classification χ_m for each mono m, the fact that $\mathrm{Sub}_{\mathscr{E}}(A)$ is thin grants us that for each χ_m there is also a unique mono m.

Put simply, $[\![-]\!]_n \circ x$ (or rather its transpose) is the character of $m_n^{(L)}$. We can define the transposed morphism by

$$\chi_{m_n^{(L)}} = \vec{\sigma}_n \mapsto \vec{\sigma}_n \in [\![x]\!]_n : \Sigma^n \longrightarrow \Omega,$$

for some state $x \in X$ and $\vec{\sigma}_n = (\sigma_1, \ldots, \sigma_n)$.

We can now express a "language" starting in x as a family of monos

$$L_x \coloneqq \left(m_n^{(L_x)} \colon [\![x]\!]_n \rightarrowtail \Sigma^n \right)_{n \in \mathbb{N}}.$$

Relating L_x to $\mathcal{EM}$-style semantics While conceivable as a intermediate step, the above does not have an immediately obvious relation to the $\mathcal{EM}$-style semantics. The issue remains representing $\Sigma^\star$ and specifically $\mathbf{P}(\Sigma^\star)$. It appears necessary to strengthen the assumptions on $\mathscr{E}$ beyond an elementary topos.

Topos with countable coproducts Adamek, et. al. discuss automata in a symmetric monoidal closed category $\mathscr{D} = (\mathscr{D}, \otimes, I)$, where here

$$\mathscr{D} = \mathscr{E}, \qquad \otimes = \times, \qquad I = 1$$

with a free monoid $X^{\circledast} = \Sigma^\star$ and a "language"

$$L : X^{\circledast} \longrightarrow Y$$

where $Y = \Omega$ describes the output. For a functor of the form $TQ = Y \times Q^X$, the terminal coalgebra is[5] $Y^{X^{\circledast}}$, which is $\Omega^{\Sigma^\star}$ in our case.

For this we require $\mathscr{E}$ to have countable coproducts, as $X^{\circledast} = \coprod_{n<\omega} X^n$, which is the initial algebra of $FQ = I + X \otimes Q$.

This provides us with the sufficient structure to define $[\![-]\!]$. For a Coalgebra $\langle e, d \rangle : X \longrightarrow \Omega \times X^\Sigma$, we can intuitively define

$$[\![x]\!] = \left\{ \vec{\sigma} \;\middle|\; e(\overline{d(x)}(\vec{\sigma})) \right\},$$

where $\overline{d(x)}$ is the canonical extension of $d\colon X \longrightarrow X^\Sigma$ over the free monoid.

The definition of a language by Adamek, et. al., would be a morphism in $\mathscr{E}$ of the form $\Sigma^\star \longrightarrow \Omega$. We can represent this internally as $\Omega^{\Sigma^\star} \cong \mathbf{P}(\Sigma^\star)$, which gives us the expected result.

Considering our previous definition, we could also describe it as a single mono (as opposed to a family of monos)

$$L'_x \colon [\![x]\!] \rightarrowtail \Sigma^\star.$$

This is not surprising, as MacLane points out that[6] a power object (or the generalised element of a power object) $1 \rightarrowtail \mathbf{P}(A)$ corresponds directly to a mono $S \rightarrowtail A$.

Recall that L_x is a family of monos. How does this relate to L'_x? Granting the existence of L_x and transitively that of countable coproducts, we want to know if

$$(L_x)_n \stackrel{?}{=} \{ \vec{\sigma} \in \mathsf{Ob}(L'_x) \mid \|\vec{\sigma}\| = n \}$$

for every $n \in \mathbb{N}$. Note as a matter of formal pedantry, that the first usage of n occurs in the meta-language, where we are indexing a family of monos, while in the second instance, n is an object in $\mathscr{E}$, that of the same type as $\|\sigma_1 \ldots \sigma_n\|$, a map from a free monoid $\sigma_1 \ldots \sigma_n$ to "n".

[5] Jiri Adamek, Stefan Milius, and Henning Urbat. *Syntactic Monoids in a Category*. 2015. arXiv: 1504.02694, p. 7.

[6] MacLane and Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, p. 165.

Topos with a natural number object If we decide that the existence of countable coproducts is too restrictive, we can consider an alternative approach, that would require $\mathscr{E}$ to express the notion of "countably", without requiring concrete countable coproducts. (The topos $\mathbf{Eff}$[7] is an example of a category with a NNO, but not arbitrarily cocomplete, specifically without countable coproducts**[citation needed]**).

A *natural number object* is[8] an object $N \in \mathsf{Ob}(\mathscr{E})$ with morphisms o, s as indicated here

$$\begin{array}{ccccc} 1 & \xrightarrow{o} & N & \xrightarrow{s} & N \\ & \searrow^{x} & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{u} & X \end{array}$$

where for any other object $X \in \mathsf{Ob}(\mathscr{E})$ and analogous morphisms pair x, u, there is unique $f\colon N \longrightarrow X$ and N is unique up to isomorphism. This should also be equivalent to the F-Algebraof the functor $FX = 1 + X$, where the structure morphism of the initial algebra is exactly $\langle o, s\rangle : N \longrightarrow 1 + N$.

In $\mathbf{Sets}$, $N = \mathbb{N}$ with $o(\cdot) = 0$ and $s(n) = n + 1$ is a NNO.[9] Every NNO is also a model of Peano arithmetic,[10]

$$n = 0 \vee \exists m.\, m = s(n)$$

$$\neg(s(n) = 0)$$

$$s(n) = s(m) \implies n = m$$

$$(0 \in P \wedge \forall n.\, (n \in P) \implies s(n) \in P) \implies P = N$$

for $n, m \in \mathsf{Ob}(N)$ and $P \in \mathsf{Ob}(\Omega^N)$.

In the internal logic, we can reason with a NNO N, just like**[citation needed]**with $\mathbb{N}$ in $\mathbf{Sets}$.

Idea: We can represent a "$\Sigma^\star$" using an object $(1 + (1 + \Sigma))^{N \times N}$. To give intuition, assume a category $\mathscr{C}$ *has* countable coproducts, allowing the direct definition of $\Sigma^\star$, for $\sigma_1 \ldots \sigma_m \in \Sigma^\star$:

$$f(\sigma_1 \ldots \sigma_m) = (n, i) \mapsto \begin{cases} \iota_1(*) & \text{if } n \neq m \\ \iota_2(\iota_1(*)) & \text{if } i > m \\ \iota_2(\iota_2(\sigma_i)) & \text{else} \end{cases}$$

Note that this allows us to map every $\Sigma^\star$ to this kind of an exponential object, but the reverse is not the case: The maps

$$(n, i) \mapsto \iota_2(\iota_2(\sigma))$$

or

$$(n, i) \mapsto \begin{cases} \iota_1(*) & \text{if } i > 0 \\ \iota_2(\iota_2(\sigma)) & \text{else} \end{cases}$$

for some fixed σ do not unambiguously correspond to a $\Sigma^\star$.

The transpose of $\ell\colon \Sigma^\star \longrightarrow (1 + (1 + \Sigma))^{N \times N}$ is $\bar{\ell}\colon \Sigma^\star \times N \longrightarrow (1 + (1 + \Sigma))^N$.

An imaginable further variation is the following

$$\bar{\ell}(n, \sigma_1 \ldots \sigma_m) = \begin{cases} \iota_1(*) & \text{if } n \neq m \\ \iota_2\left(i \mapsto \begin{cases} \iota_1(*) & \text{if } 1 \leq i \leq m \\ \iota_2(\sigma_i) & \text{else} \end{cases}\right) & \text{else} \end{cases}.$$

of the type $\bar{\ell}\colon N \times \Sigma^\star \longrightarrow 1 + (1 + \Sigma)^N$.

We would like to demonstrate $F := (1 + (1 + \Sigma))^{N \times N}$ this can serve as the carrier for the terminal coalgebra, which should also grant us that if $\mathscr{E}$ had to countable coproducts, that the following would commute:

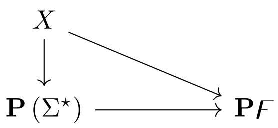
---

[7] J.M.E. Hyland. "The Effective Topos". In: *The L. E. J. Brouwer Centenary Symposium*. Ed. by A.S. Troelstra and D. van Dalen. Vol. 110. Studies in Logic and the Foundations of Mathematics. Elsevier, 1982, pp. 165–216. DOI: https://doi.org/10.1016/S0049-237X(09)70129-6. URL: https://www.sciencedirect.com/science/article/pii/S0049237X09701296.

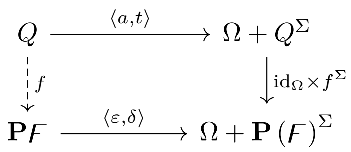
[8] Peter T Johnstone. *Topos theory*. Courier Corporation, 2014, p. 165.

[9] Francis Borceux. *Handbook of Categorical Algebra: Volume 3, Sheaf Theory*. Vol. 3. Cambridge university press, 1994, p. 455.

[10] Ibid., p. 457, p. 456.



To prove that $\mathcal{F}$ is a carrier for the terminal coalgebara, we need a unique coalgebra homomorphism $f\colon X \longrightarrow \mathbf{P}\mathcal{F}$:



We can consider the two components separately:

$$\varepsilon \circ f = a\colon Q \longrightarrow \Omega \qquad \text{(termination)}$$
$$\delta \circ f = t \circ f^{\Sigma}\colon Q \longrightarrow \Omega \qquad \text{(transition)}$$

For (termination), we need to ensure that if the current state is accepting ($a\colon Q \longrightarrow \Omega$), then the "empty word" is also accepted:

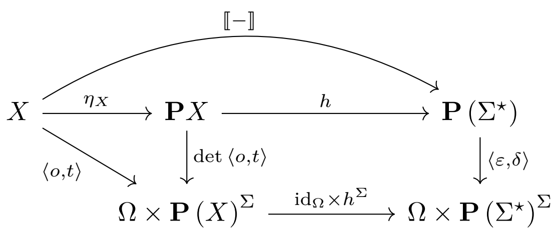
$$\exists f \in \mathbf{P}\mathcal{F} \forall n \forall i.\, f(n,i) = \pi_1(*)$$

For (transition), we need to ensure that the addition of a $\sigma \in \mathsf{Ob}(\Sigma)$ properly extends the accepted words:

$$\forall \sigma \in \Sigma$$

... The preceding investigation was suddenly interrupted and possibly deferred to a later point in time ...

Suitability of $[\![-]\!]$ Assuming $\mathscr{E}$ is countably extensive, we have the following situation,



where for a given coalgebra $\langle o,t\rangle : X \longrightarrow \Omega \times X^{\Sigma}$, we define

$$[\![x]\!] \coloneqq \left\{ \vec{\sigma} \in \Sigma^{\star} \;\middle|\; o(\overline{t(x)}(\vec{\sigma})) \right\},$$
$$\eta_X(x) \coloneqq \{\, y \mid y = x \,\},$$
$$h(\mathfrak{X}) \coloneqq \left\{ \vec{\sigma} \in \Sigma^{\star} \;\middle|\; \exists\, x \in \mathfrak{X}.\, o(\overline{t(x)}(\vec{\sigma})) \right\},$$
$$\varepsilon(L) \coloneqq \epsilon \in L$$
$$\delta(L) \coloneqq \sigma \mapsto \{\, \vec{\sigma} \in \Sigma^{\star} \mid \sigma \cdot \vec{\sigma} \in L \,\}$$
$$\pi_1(\det \langle o,t\rangle) \coloneqq \mathfrak{X} \mapsto \exists\, x \in \mathfrak{X}.\, o(x)$$
$$\pi_2(\det \langle o,t\rangle) \coloneqq \mathfrak{X} \mapsto \left(\sigma \mapsto \bigcup_{x \in \mathfrak{X}} t(x)(\sigma) \right)$$

where the last four definitions follow Silva, et. al.[11] in spirit. To verify, that our definition of $[\![-]\!]$ is sensible, we analyse the two commuting polygons in the internal logic:

$$x\colon X \vdash \langle o,t\rangle = \det \langle o,t\rangle \circ \eta_X$$

[11] Alexandra Silva et al. "Generalizing determinization from automata to coalgebras". In: *Logical Methods in Computer Science* 9 (2013), p. 5.

and

$$\mathfrak{X} \colon \mathbf{P}X, \sigma \colon \Sigma \vdash \langle \varepsilon, \delta \rangle \circ h = \mathrm{id}_\Sigma \times h^\Sigma \circ \det \langle o, t \rangle \,,$$

where we can split the latter equation into two

$$\mathfrak{X} \colon \mathbf{P}X \vdash \varepsilon \circ h \iff \pi_1(\det \langle o, t \rangle),$$

and

$$\mathfrak{X} \colon \mathbf{P}X, \sigma \colon \Sigma \vdash \delta \circ h = h^\Sigma \circ \pi_2(\det \langle o, t \rangle).$$

Singleton Determinisation Verify,

$$\begin{aligned}
&\vdash \langle o, t \rangle = \det \langle o, t \rangle \circ \eta_X \\
x \colon X &\vdash \langle o, t \rangle (x) = \det \langle o, t \rangle (\eta_X(x)) \\
x \colon X &\vdash \langle o, t \rangle (x) = \left\langle \mathfrak{X} \mapsto \exists x \in X.\, o(x), \mathfrak{X} \mapsto \left(\sigma \mapsto \bigcup_{x \in \mathfrak{X}} t(x)(\sigma) \right) \right\rangle (\eta_X(x)) \\
x \colon X &\vdash \langle o, t \rangle (x) = \left\langle \exists x \in \eta_X(x).\, o(x), \left(\sigma \mapsto \bigcup_{x \in \eta_X(x)} t(x)(\sigma) \right) \right\rangle \\
x \colon X &\vdash \langle o, t \rangle (x) = \langle o(x), (\sigma \mapsto t(x)(\sigma)) \rangle \\
x \colon X &\vdash \langle o, t \rangle (x) = \langle o(x), t(x) \rangle \\
x \colon X &\vdash \langle o, t \rangle (x) = \langle o, t \rangle (x) \\
&\vdash \langle o, t \rangle = \langle o, t \rangle
\end{aligned}$$

■

Termination of the Terminal Coalgebra Verify,

$$\begin{aligned}
&\vdash \varepsilon \circ h \iff \pi_1(\det \langle o, t \rangle) \\
\mathfrak{X} \colon \mathbf{P}X &\vdash \varepsilon(h(\mathfrak{X})) \iff (X \mapsto \exists x \in X.\, o(x))(\mathfrak{X}) \\
\mathfrak{X} \colon \mathbf{P}X &\vdash \epsilon \in \left\{ \vec{\sigma} \;\middle|\; \exists x \in \mathfrak{X}.\, o(\overline{t(x)}(\vec{\sigma})) \right\} \iff \exists x \in \mathfrak{X}.\, o(x) \\
\mathfrak{X} \colon \mathbf{P}X &\vdash \exists x \in \mathfrak{X}.\, o(\overline{t(x)}(\epsilon)) \iff \exists x \in \mathfrak{X}.\, o(x) \\
\mathfrak{X} \colon \mathbf{P}X &\vdash \exists x \in \mathfrak{X}.\, o(x) \iff \exists x \in \mathfrak{X}.\, o(x)
\end{aligned}$$

■

Transitions of the Terminal Coalgebra Verify with context $\mathfrak{X} \colon \mathbf{P}X, \sigma \colon \Sigma$,

$$\begin{aligned}
&\vdash \delta \circ h = h^\Sigma \circ \pi_2(\det \langle o, t \rangle) \\
&\vdash \delta(h(\mathfrak{X})) = h^\Sigma(\pi_2(\det \langle o, t \rangle)(\mathfrak{X})) \\
&\vdash \delta(h(\mathfrak{X})) = h^\Sigma \left(\sigma \mapsto \bigcup_{y \in \mathfrak{X}} t(y)(\sigma) \right) \\
&\vdash \delta(h(\mathfrak{X})) = \sigma \mapsto h \left(\bigcup_{y \in \mathfrak{X}} t(y)(\sigma) \right) \\
&\vdash \sigma \mapsto \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; \sigma \cdot \vec{\sigma} \in \left\{ \vec{\sigma} \;\middle|\; \exists x \in \mathfrak{X}.\, o(\overline{t(x)}(\vec{\sigma})) \right\} \right\} = \dots \\
&\vdash \sigma \mapsto \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; \exists x \in \mathfrak{X}.\, o(\overline{t(x)}(\sigma \cdot \vec{\sigma})) \right\} \\
&= \sigma \mapsto \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; \exists x \in \left(\bigcup_{y \in \mathfrak{X}} t(y)(\sigma) \right).\, o(\overline{t(x)}(\vec{\sigma})) \right\} \\
&\vdash \left\{ \vec{\sigma} \;\middle|\; \exists x \in \mathfrak{X}.\, o(\overline{t(x)}(\sigma \cdot \vec{\sigma})) \right\} = \left\{ \vec{\sigma} \;\middle|\; \exists x \in \mathfrak{X}.\, o(\overline{t(x)}\,(\sigma \cdot \vec{\sigma})) \right\}
\end{aligned}$$

where we can legitimate the inference step

$$\exists x \in \left(\bigcup_{y \in \mathfrak{X}} t(y)(\sigma) \right) . o(\overline{t(x)}(\vec{\sigma})) \iff \exists x \in \mathfrak{X}. o(\overline{t(x)}(\sigma \cdot \vec{\sigma}))$$

will be legitimated below.

Verification of $[\![-]\!]$ As a final step, we have to ensure that for a $x \colon X$ the following holds:

$$\begin{aligned} &x \colon X \vdash [\![x]\!] = h(\eta_X(x)) \\ &x \colon X \vdash \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(x)}(\vec{\sigma})) \right\} = \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; \exists x \in \eta_X(x). o(\overline{t(x)}(\vec{\sigma})) \right\} \\ &x \colon X \vdash \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(x)}(\vec{\sigma})) \right\} = \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(x)}(\vec{\sigma})) \right\} \end{aligned} \qquad \blacksquare$$

This gives us a satisfactory conclusion regarding the suitability of $[\![-]\!]$ in terms of the internal logic of $\mathscr{E}$ to express the semantics of a non-deterministic automaton described by $\langle o, t \rangle$.

The canonical extension of f on a (free) monoid $\Sigma^\star$ As a final point of clarification, it is necessary to consider the definition and properties of

$$\overline{t(x)}(\vec{\sigma}) = \begin{cases} \{x\} & \text{if } \vec{\sigma} = \epsilon \\ \bigcup_{x' \in t(x)(\sigma)} \overline{t(x')}(\vec{\sigma}') & \text{if } \vec{\sigma} = \sigma \cdot \vec{\sigma}' \end{cases}$$

for a $t \colon X \longrightarrow \mathbf{P}X^\Sigma$, $x \colon X$ and $\vec{\sigma} \colon \Sigma^\star$. Keep in mind that this is *not* a definition. We instead have to demonstrate that a morphism exists with properties like these when considered point-wise.

Note that $\Sigma^\star$ is the initial algebra of the functor $FX = 1 + \Sigma \times X$, meaning we have a have a unique morphism $h(n,c) \colon \Sigma^\star \longrightarrow \mathbf{P}\,(X)^X$, for which

$$\begin{array}{ccc} 1 + \Sigma \times \Sigma^\star & \xrightarrow{[\mathrm{nil};\mathrm{cons}]} & \Sigma^\star \\ \Big\downarrow{\scriptstyle \mathrm{id}_1 + \mathrm{id}_\Sigma \times h(n,c)} & & \Big\downarrow{\scriptstyle h(n,c)} \\ 1 + \Sigma \times \mathbf{P}\,(X)^X & \xrightarrow{[n;c]} & \mathbf{P}\,(X)^X \end{array}$$

commutes. In this context, we want $h(n,c)$ to denotes the function generated by a word $\vec{\sigma}$, such that

$$h(n,c)(\vec{\sigma}) = x \mapsto \overline{t(x)}(\vec{\sigma}),$$

holds for an arbitrary $\vec{\sigma}$.

We have to define a

$$n \colon 1 \longrightarrow \mathbf{P}\,(X)^X,$$

$$c \colon \Sigma^\star \times \mathbf{P}\,(X)^X \longrightarrow \mathbf{P}\,(X)^X$$

and the dependent

$$h(n,c) \colon \Sigma^\star \longrightarrow \mathbf{P}\,(X)^X$$

to demonstrate that the above diagram commutes. We can split this up into two equations:

$$n \circ \mathrm{id}_1 = h \circ \mathrm{nil} \tag{1}$$

$$c \circ (\mathrm{id}_\Sigma \times h) = h \circ \mathrm{cons} \tag{2}$$

and assume the definitions:

$$\begin{aligned} n(*) &:= x \mapsto \{x\} = \eta_x \\ c(\sigma, f) &:= x \mapsto \bigcup_{x' \in t(x)(\sigma)} f(x') \\ h(n,c)(\vec{\sigma}) &:= \begin{cases} n & \text{if } \vec{\sigma} = \epsilon \\ c(\sigma, h(n,c)(\vec{\sigma}')) & \text{if } \vec{\sigma} = \sigma \cdot \vec{\sigma}' \end{cases} \end{aligned}$$

Note the implicit usage of the transition morphism t in the definition of c.

Commutativity using $h(n,c)$ First consider the equation involving nil,

$$\begin{aligned} &\vdash & n &= h(n,c)\circ \mathrm{nil} \\ &\vdash & n(*) &= h(n,c)(\mathrm{nil}(*)) \\ &\vdash & n &= n \qquad \blacksquare \end{aligned}$$

and for the "cons"-path:

$$\begin{aligned} &\vdash & c\circ(\mathrm{id}_\Sigma \times h(n,c)) &= h(n,c)\circ \mathrm{cons} \\ \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & c\circ(\mathrm{id}_\Sigma \times h(n,c))(\sigma,\vec{\sigma}) &= h(n,c)\circ \mathrm{cons}(\sigma,\vec{\sigma}) \\ \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & c(\sigma,h(n,c)(\vec{\sigma})) &= h(n,c)(\mathrm{cons}(\sigma,\vec{\sigma})) \\ \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & c(\sigma,h(n,c)(\vec{\sigma})) &= c(\sigma,h(n,c)(\vec{\sigma})) \qquad \blacksquare \end{aligned}$$

Definition of t in relation to $h(n,c)$ It is clear that $\overline{t(-)}\colon X\times\Sigma^\star \longrightarrow \mathbf{P}X$ is the exponential transposition of $h(n,c)\colon \Sigma^\star \longrightarrow \mathbf{P}(X)^X$, so the question remains if this satisfies the conditions we expect. Therefore, we will consider the two "constructors" of a $\Sigma^\star$ inductively. An empty-word, i.e. the base-case,

$$\begin{aligned} &\vdash & h(n,c)(\mathrm{nil}) &= x\mapsto t(\mathrm{nil})(x) = t(\epsilon)(x) \\ &\vdash & n &= x\mapsto \{x\} \\ &\vdash & \eta_X &= \eta_X \qquad \blacksquare \end{aligned}$$

and for a non-empty word, with the induction hypothesis $h(n,c)(\vec{\sigma}) = x\mapsto \overline{t(x)}(\vec{\sigma})$,

$$\begin{aligned} \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & h(n,c)(\mathrm{cons}(\sigma,\vec{\sigma})) &= (x\mapsto t(\mathrm{cons}(\sigma,\vec{\sigma}))(x)) = (x\mapsto t(\sigma\cdot\vec{\sigma})(x)) \\ \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & c(\sigma,h(n,c)(\vec{\sigma})) &= \left(x\mapsto \bigcup_{x'\in t(x)(\sigma)} \overline{t(x')}(\vec{\sigma})\right) \qquad (\texttt{apply}\ \text{I.H.}) \\ \sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash & \left(x\mapsto \bigcup_{x'\in t(x)(\sigma)} h(n,c)(\vec{\sigma})(x')\right) &= \left(x\mapsto \bigcup_{x'\in t(x)(\sigma)} h(n,c)(\vec{\sigma})(x')\right) \qquad \blacksquare \end{aligned}$$

Verifying the intended usage As a reminder, the intention was to ensure that equivalences like

$$\mathfrak{X}\colon \mathbf{P}X,\sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star \vdash \exists x\in\left(\bigcup_{y\in\mathfrak{X}} t(y)(\sigma)\right).\, o\left(\overline{t(x)}(\vec{\sigma})\right) \iff \exists x\in\mathfrak{X}.\, o\left(\overline{t(x)}(\sigma\cdot\vec{\sigma})\right)$$

or more concretely/simply

$$x\colon X,\sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star \vdash \exists x'\in t(x)(\sigma).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff o\left(\overline{t(x)}(\sigma\cdot\vec{\sigma})\right).$$

In fact, we might regard the former as a special case of the latter, where the $\mathfrak{X}$ are the states following a transition from a x over some $\tilde{\sigma}\colon\Sigma$:

$$\begin{aligned} x\colon X,\sigma\colon\Sigma,\tilde{\sigma}\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash \exists x'\in\left(\bigcup_{y\in t(x)(\tilde{\sigma})} t(y)(\sigma)\right).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff \exists x'\in t(x)(\tilde{\sigma}).\, o\left(\overline{t(x')}(\sigma\cdot\vec{\sigma})\right) \\ x\colon X,\sigma\colon\Sigma,\tilde{\sigma}\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash \exists x'\in t(x)(\tilde{\sigma}\cdot\sigma).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff \exists x'\in t(x)(\tilde{\sigma}).\, o\left(\overline{t(x')}(\sigma\cdot\vec{\sigma})\right) \end{aligned}$$

(We can legitimate this claim in general, by extending the automaton by a fresh $\tilde{\sigma}$ that maps x to $\mathfrak{X}$, and that wouldn't affect any transitions beyond that.)

So restricting our attention to the latter formula,

$$\begin{aligned} x\colon X,\sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash \exists x'\in t(x)(\sigma).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff o\left(\overline{t(x)}(\sigma\cdot\vec{\sigma})\right) \\ x\colon X,\sigma\colon\Sigma,\vec{\sigma}\colon\Sigma^\star &\vdash \exists x'\in t(x)(\sigma).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff o\left(\bigcup_{x'\in t(x)(\sigma)} \overline{t(x')}(\vec{\sigma})\right) \end{aligned}$$

Reminding ourselves that o is a "$\exists$-style" check, and that for an arbitrary non-deterministic state $\mathfrak{X}$

$$o(\mathfrak{X}) \iff \exists\, x_i \in \mathfrak{X}.\, o\left(\{x_i\}\right)$$

holds. Therefore,

$$x \colon X, \sigma \colon \Sigma, \vec{\sigma} \colon \Sigma^\star \vdash \qquad \exists\, x' \in t(x)(\sigma).\, o\left(\overline{t(x')}(\vec{\sigma})\right) \iff \exists\, x \in \bigcup_{x' \in t(x)(\sigma)} \overline{t(x')}(\vec{\sigma}).\, o\left(\{x\}\right)$$

TODO