A formulation of ρ in a Topos $\mathscr{E}$

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We know that $\rho : TF \Rightarrow TF$, where the monad $TX = \wp(X)$ and an endofunctor $FX = 2 \times \wp(X)$ map a set of deterministic automata into a single non-deterministic automaton N, where N accepts a word if any deterministic automaton would accept it and state transitions merge all deterministic transitions.

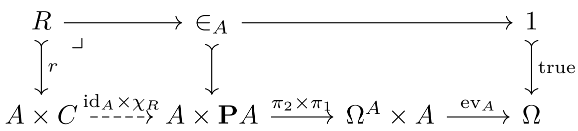
The above occurs in $\mathbf{Set}$. Can we translate this into a topos, where

$$TX = \mathbf{P}X$$

$$FX = \Omega \times X^{\Sigma}$$

To this end, we have to define the "power object functor" $\mathbf{P}-$, the "exponential functor" $-^A$ and the "product functor" $- \times A$ for some $A \in \mathsf{Ob}(\mathscr{E})$ (the latter two, which are given in Cartesian closed category (terminal, product, exponential), are known to be adjunct).

Definition of a Topos For a category $\mathscr{E}$, we speak of a *power object* $A \in \mathsf{Ob}(\mathscr{E})$ as an object $\mathbf{P}A = \Omega^A \in \mathsf{Ob}(\mathscr{E})$ along with a morphism $\in_A \rightarrowtail A \times \mathbf{P}A$, when for every $C \in \mathsf{Ob}(\mathscr{E})$ and mono $R \longrightarrow A \times C$ the following commutes (composing diagrams by McLarty,[1] Johnstone[2],[3] Caramello[4] and from nLab[5]):



with a unique $\chi_R : C \longrightarrow \mathbf{P}A$ and R being the pullback.

If $\mathscr{E}$ with all finite limits has power objects for all objects $\mathsf{Ob}(\mathscr{E})$, then we call $\mathscr{E}$ a *(elementary) topos*.[6] The qualifier "elementary" distinguishes the notion

[1] Colin McLarty. *Elementary categories, elementary toposes*. Clarendon Press, 1992, p. 120.

[2] Peter T Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford University Press, 2002, p. 86.

[3] Peter T Johnstone. *Topos theory*. Courier Corporation, 2014, p. 43.

[4] Olivia Caramello and Riccardo Zanfa. *On the dependent product in toposes*. 2019. arXiv: 1908.08488 [math.CT]. URL: https://arxiv.org/abs/1908.08488, p. 5.

[5] nLab authors. *power object*. https://ncatlab.org/nlab/show/power+object. Revision 8. May 2024.

[6] Michael Barr and Charles Wells. *Toposes, triples, and theories*. Springer-Verlag, 2000, p. 63.

from *Grothendieck topos*, which are a special instance of elementary toposes.[7]

From the above, we can derive arbitrary finite co limits[8] and exponential objects B^A,[9] such that for all $g : Z \times A \longrightarrow B$, there is a unique $f : Z \longrightarrow B^A$ in

$$\begin{array}{ccc} Z \times A & & \\ {\scriptstyle f\times \mathrm{id}_A}\Big\downarrow & \searrow^{g} & \\ B^A \times A & \xrightarrow{\mathrm{ev}_{A,B}} & B \end{array}$$

The subobject classifier,

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \mathrm{true}} \\ B & \xrightarrow{\;\varphi\;} & \Omega \end{array}$$

commutes, follows from $\Omega = \Omega^1 = \mathbf{P}1$.

There are multiple equivalent definitions,[10] for example MacLane[11] postulate all pullbacks and a terminal objects (which amount's to $\mathscr{E}$ being complete), the subobject classifier Ω and then describes power objects PA along with a morphism $\in_A: A \times \mathbf{P}A \longrightarrow \Omega$, such that for every $f : A \times B \longrightarrow \Omega$ there is a unique arrow $g : B \longrightarrow \mathbf{P}A$ and

$$\begin{array}{ccc} B \times A & \xrightarrow{\quad f \quad} & \Omega \\ {\scriptstyle g\times \mathrm{id}_A}\Big\downarrow & & \Big\| \\ \mathbf{P}A \times A & \xrightarrow{\quad \in_A \quad} & \Omega \end{array}$$

where the morphism $\in_A = \mathrm{ev}_{A,\Omega}$ is not to be confused with the object $\in_A$ given above. Taken as a contravariant functor, $\mathbf{P}- : \mathscr{E} \longrightarrow \mathscr{E}$ maps an object in $\mathscr{E}$ to its respective power object. A morphism $h : A \longrightarrow B$ is raised to $\mathbf{P}h : \mathbf{P}B \longrightarrow \mathbf{P}A$, so that

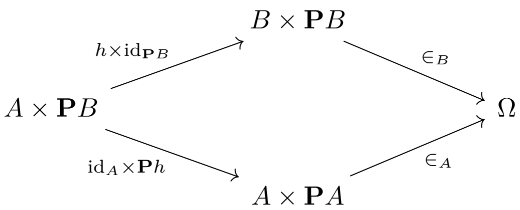
[7] Johnstone, *Topos theory*, p. 24.

[8] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012, p. 180.

[9] Ibid., p. 167.

[10] Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, p. vii.

[11] MacLane and Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, p. 163.



commutes.

Example in Set The power object of any set A is $\wp(A) := \{B \mid B \subseteq A\}$, exponential objects B^A are set of functions of type $A \longrightarrow B$ and the sub-object classifier is $\mathbf{P}1 = \wp(\{*\}) = \{\{*\}, \{\}\} \cong 2 \cong \{\top, \bot\}$ and φ is the characteristic function indicating if a an element of a (super-)set S is part of a subset B. We can interpret McLarty's $\in_A$ as the subset

$$\in_A := \{\langle a, X\rangle \mid a \in X\} \subseteq A \times \Omega^A$$

where "$\in$" is the usual set-theoretical membership relation.

Constituent Functors We will be using the **co**variant Power-Object Functor, as this is necessary for the Coalgebra to be defined on an Endofunctor. As expected, the "binary product functor $- \times A$ with a fixed object $A \in \mathrm{Ob}(\mathscr{E})$" maps $B \in \mathrm{Ob}(\mathscr{E})$ to $A \times B \in \mathrm{Ob}(\mathscr{E})$, and maps a morphism $m : B \longrightarrow C$ to a morphism $f \times A : B \times A \longrightarrow C \times A$. The "exponential functor $-^A$ with a fixed domain $A \in \mathrm{Ob}(\mathscr{E})$" maps a $B \in \mathrm{Ob}(\mathscr{E})$ and a morphism $f : B \longrightarrow C$ to $f^A : B^A \longrightarrow C^A$ so that

$$\begin{array}{ccc} B^A \times A & \xrightarrow{\mathrm{ev}_{A,B}} & B \\ {\scriptstyle f^A \times \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle f} \\ C^A \times A & \xrightarrow{\mathrm{ev}_{A,C}} & C \end{array}$$

commutes.

Defining ρ in $\mathscr{E}$ Recall that in **Set**, Jacobs, et. al. define[12] $\rho_X = \rho_{X1} \times \rho_{X2} : \wp\left(2 \times X^\Sigma\right) \longrightarrow 2 \times \wp(X)^\Sigma$ component-wise,

$$\rho_1(U) = 1 \iff \exists h \in X.\, \langle 1, h\rangle \in U$$

and

$$x = \rho_2(U)(a) \iff \exists \langle b, h\rangle \in U.\, h(a) = x.$$

This now becomes $\varrho_X : \mathbf{P}(\Omega \times X^\Sigma) \longrightarrow \Omega \times \mathbf{P}X^\Sigma$.

[12] Bart Jacobs, Alexandra Silva, and Ana Sokolova. "Trace semantics via determinization". In: *International Workshop on Coalgebraic Methods in Computer Science*. Springer. 2012, pp. 109–129, p. 117.

Here the question arises, what a power-object of a sub-object classifier might be? Likewise, how does the power-object behave over products and exponential objects? Back in $\mathbf{Set}$, we could make use of properties like

$$\wp(1+\Sigma\times X)\cong 2\times\wp(\Sigma\times X)\cong 2\times\wp(X)^{\Sigma},$$

as

$$2^{1+\Sigma\times X}\cong 2\times 2^{\Sigma\times X}\cong 2\times 2^{X^{\Sigma}}.$$

Reminding ourselves that $\mathbf{P}A\cong\Omega^{A}$, we can make use of properties enjoyed by exponential objects,[13] such as transposition (currying)

$$\mathrm{Hom}_{\mathscr{C}}(A,C^{B})\cong\mathrm{Hom}_{\mathscr{C}}(B\times A,C).$$

As a Product UMP It is clear, that $\Omega\times\mathbf{P}X^{\Sigma}$ has two projections

$$\pi_1:\Omega\times\mathbf{P}X^{\Sigma}\longrightarrow\Omega\qquad\pi_2:\Omega\times\mathbf{P}X^{\Sigma}\longrightarrow\mathbf{P}X^{\Sigma}$$

that constitute a universal cone. If we can provide two further morphisms

$$\rho_1:\mathbf{P}\left(\Omega\times X^{\Sigma}\right)\longrightarrow\Omega\qquad\rho_2:\mathbf{P}\left(\Omega\times X^{\Sigma}\right)\longrightarrow\mathbf{P}X^{\Sigma}$$

then the universal property of products gives us a unique morphism, which we shall already conveniently refer to as

$$\rho_X:\mathbf{P}\left(\Omega\times X^{\Sigma}\right)\dashrightarrow\Omega\times\mathbf{P}X^{\Sigma}.$$

Here's an idea: The cone-morphisms ρ_1 and ρ_2 will respectively be defined as

$$\rho_1:\mathbf{P}\left(\Omega\times X^{\Sigma}\right)\longrightarrow\mathbf{P}\Omega\longrightarrow\Omega\qquad\rho_2:\mathbf{P}\left(\Omega\times X^{\Sigma}\right)\longrightarrow\mathbf{P}X^{\Sigma}\longrightarrow\mathbf{P}X^{\Sigma}.$$

Subobject of a Power-Object-Product These are simply

$$\mathbf{P}\pi_1:\mathbf{P}(A\times B)\longrightarrow\mathbf{P}A,$$

and

$$\mathbf{P}\pi_2:\mathbf{P}(A\times B)\longrightarrow\mathbf{P}B,$$

as $\mathbf{P}-$ is covariant.

Elaborating ρ_1 and ρ_2 Given $\mathbf{P}\pi_1$ and $\mathbf{P}\pi_2$, the constructing the cone from $\mathbf{P}\left(\Omega\times X^{\Sigma}\right)$ requires two further morphisms, of the forms

$$\mathbf{P}\Omega\longrightarrow\Omega\qquad\text{and}\qquad\mathbf{P}\left(X^{\Sigma}\right)\longrightarrow\mathbf{P}X^{\Sigma}.$$

The the former, consider the subobject,

$$\begin{array}{ccc}\{X\mid\exists x\in X.\,x\} & \longrightarrow & 1\\ \big\downarrow f & & \big\downarrow \mathrm{true}\\ \mathbf{P}\Omega & \xrightarrow{\ \chi_f\ } & \Omega\end{array}$$

[13]Steve Awodey. *Category Theory*. Oxford, England: Oxford University Press, 2006, p. 119.

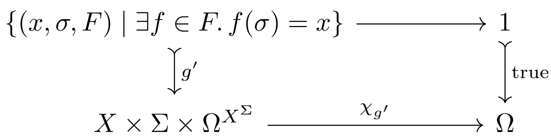
For the latter, consider

$$\begin{aligned}
&\mathbf{P}\left(X^{\Sigma}\right) \longrightarrow (\mathbf{P}X)^{\Sigma} : g \\
&\cong \Omega^{X^{\Sigma}} \longrightarrow \left(\Omega^{X}\right)^{\Sigma} \\
&\cong \Omega^{X^{\Sigma}} \longrightarrow \Omega^{\Sigma\times X} \\
&\cong \Sigma \times X \times \Omega^{X^{\Sigma}} \longrightarrow \Omega && \text{(curry)} \\
&\cong \Sigma \times X \times \mathbf{P}\left(X^{\Sigma}\right) \longrightarrow \Omega : g'
\end{aligned}$$

We can regard the last form as a characteristic morphism of the subobject "containing", taking the liberty of thinking in $\mathbf{Set}$,

> *All $x \in X$, $\sigma \in \Sigma$ and $F \in \mathbf{P}\left(X^{\Sigma}\right)$ (that is to say $F \subseteq X^{\Sigma}$) where there exists a $f \in F$, such that $f(\sigma) = x$.*

or put in terms of the internal logic of $\mathscr{E}$,



It would be worthwhile to translate these internal formulations back into morphisms of $\mathscr{E}$.

These results gives us

$$\rho_1 = \chi_f \circ \mathbf{P}\pi_1 : \mathbf{P}\left(\Omega \times X^{\Sigma}\right) \longrightarrow \Omega$$

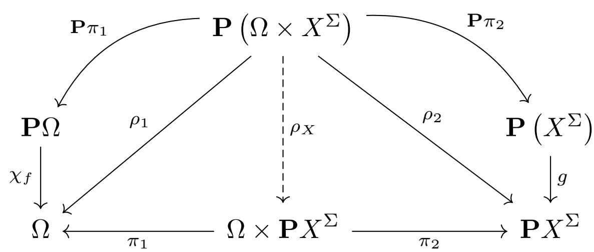
and

$$\rho_2 = g \circ \mathbf{P}\pi_2 : \mathbf{P}\left(\Omega \times X^{\Sigma}\right) \longrightarrow \mathbf{P}X^{\Sigma}.$$

Due to the uniqueness of ρ_X, we can conclude that the above construction gives us a concrete definition:

$$\rho_X = (\chi_f \times g) \circ \langle \mathbf{P}\pi_1, \mathbf{P}\pi_2 \rangle .$$

Overview and Review of the Construction The following commutative diagram summarises the construction



Before proceeding to check if this satisfies the conditions of the $\mathcal{EM}$-distributive law, I would like to verify if the arrows make sense in terms of toposes as generalised sets:

- The projection ρ_1, itself a characteristic morphism of the subobject that "contains" at least one accepting automaton.
- Going by MacLane,[14] we know that subobjects

  $$m : S \rightarrowtail A$$

  may also be described as

  $$s : 1 \longrightarrow \mathbf{P}A.$$

  The subobject corresponding to $\mathbf{P}\left(X^{\Sigma}\right)$ denotes state-transitions, that collectively step to a sub-object $\mathbf{P}X$ via some $\sigma \in \Sigma$, which we precisely describe using $\mathbf{P}X^{\Sigma}$.

This intuitively matches the above mentioned description by Jacobs, et. al..

Verifying the Distributivity Laws Recall that $FX = \Omega \times X^{\Sigma}$. It remains to verify if a "singleton" power-object distributes to a non-deterministic automaton over a single state,

$$\begin{array}{ccc} & \Omega \times X^{\Sigma} & \\ {\scriptstyle \eta_{FX}} \swarrow & & \searrow {\scriptstyle F(\eta_X)} \\ \mathbf{P}\left(\Omega \times X^{\Sigma}\right) & \xrightarrow{\quad \rho_X \quad} & \Omega \times \mathbf{P}X^{\Sigma} \end{array}$$

and if "flattening" power-objects of automata and of states distribute well as well,

$$\begin{array}{ccccc} \mathbf{P}\left(\mathbf{P}\left(\Omega \times X^{\Sigma}\right)\right) & \xrightarrow{\mathbf{P}(\rho_X)} & \mathbf{P}\left(\Omega \times \mathbf{P}X^{\Sigma}\right) & \xrightarrow{\rho_{\mathbf{P}X}} & \Omega \times \mathbf{P}\mathbf{P}X^{\Sigma} \\ \downarrow {\scriptstyle \mu_{FX}} & & & & \downarrow {\scriptstyle F(\mu_X)} \\ \mathbf{P}\left(\Omega \times X^{\Sigma}\right) & & \xrightarrow{\qquad \rho_X \qquad} & & \Omega \times \mathbf{P}X^{\Sigma} \end{array}$$

The Power-Object-Functor is a Monad First we have to define our terms, and make the unit η and multiplication μ of the monad explicit.

Following a comment by Zhen Lin on Stack Exchange[15], we can define η as the characteristic morphism of the transpose of the diagonal

$$\chi_{\Delta} : X \times X \longrightarrow \Omega$$

as

$$\eta_X : X \longrightarrow \Omega^X \cong \mathbf{P}X,$$

[14] MacLane and Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, p. 165.

[15] https://math.stackexchange.com/a/1192948

or in using internal logic

$$\eta_X(e) = \{x \mid e = x\}.$$

Now, whenever we encounter the a $\vdash a \in \eta_X(e)$, we know this to be equivalent to $\vdash a = e$.

Zhen gives the definition of multiplication directly using internal logic

$$\mu_X(t) = \{x \mid \exists s : \mathbf{P}X.\, x \in s \wedge s \in t\}.$$

Let us use the opportunity the rephrase ρ_1 and ρ_2 directly and point-wise in terms of the internal logic of $\mathscr{E}$ and a fixed state space $X \in \mathsf{Ob}(\mathscr{E})$:

$$\rho_1\left(A : \mathbf{P}\left(\Omega \times X^{\Sigma}\right)\right) = \exists \langle \varepsilon, \delta \rangle \in A.\, \varepsilon$$

and

$$\rho_2\left(A : \mathbf{P}\left(\Omega \times X^{\Sigma}\right)\right) = \sigma \mapsto \{x : \mathbf{P}X \mid \exists \langle \varepsilon, \delta \rangle \in A.\, \delta(\sigma) = x\}$$

so together

$$\rho_X(A) = \langle \exists \langle \varepsilon, \delta \rangle \in A.\, \varepsilon, \sigma \mapsto \{x : \mathbf{P}X \mid \exists \langle \varepsilon, \delta \rangle \in A.\, \delta(\sigma) = x\}\rangle$$

Distributitivty of the Unit In the internal logic, the first diagram reads as

$$\models \quad \rho_X \circ \eta_{FX} = F(\eta_X)$$

$$a \models \quad \rho_X(\eta_{FX}(a)) = F(\eta_X)(a)$$

at which point we can split the equation into the two cases

$$a \models \quad \rho_1(\eta_{FX}(a)) = \mathrm{id}_\Omega(\pi_1(a)) \qquad \text{(left)}$$

$$a \models \quad \rho_2(\eta_{FX}(a)) = (\eta_X)^{\Sigma}(\pi_2(a)) \qquad \text{(right)}$$

considering the simpler (left) case first,

$$a \models \quad \exists \langle \varepsilon, \delta \rangle \in \eta_{FX}(a).\, \varepsilon = \pi_1(a)$$

$$a \models \quad \exists \langle \varepsilon, \delta \rangle \in \{y \mid y = a\} \wedge \varepsilon = \pi_1(a)$$

$$a \models \quad \exists \langle \varepsilon, \delta \rangle . \langle \varepsilon, \delta \rangle = a \wedge \varepsilon = \pi_1(a)$$

given that a is a $\Omega \times X^{\Sigma}$, we can replace

$$\varepsilon, \delta \models \quad (\exists \langle \varepsilon', \delta' \rangle . \langle \varepsilon', \delta' \rangle = \langle \varepsilon, \delta \rangle \wedge \varepsilon') = \pi_1(\langle \varepsilon, \delta \rangle)$$

$$\varepsilon \models \quad \varepsilon = \varepsilon$$

and then the right case, by extending both sides with a $\sigma \in \Sigma$,

$$a, \sigma \models \quad \rho_2(\eta_{FX}(a))(\sigma) = (\eta_X)^{\Sigma}(\pi_2(a))(\sigma)$$

$$a, \sigma, z \models \quad z \in \rho_2(\eta_{FX}(a))(\sigma) \iff z \in ((\eta_X)^{\Sigma}(\pi_2(a)))(\sigma)$$

considering the left hand side of the implication, we get

$$\langle \varepsilon, \delta \rangle \in \{y \mid y = a\} \wedge \delta(\sigma) = z$$

or

$$\langle \varepsilon, \delta \rangle = a \wedge \delta(\sigma) = z$$

or

$$(\pi_2(a))(\sigma) = z,$$

while the right hand side gives us

$$z \in ((g \mapsto \eta_X \circ g)(\pi_2(a)))(\sigma)$$

or

$$z \in (\eta_X \circ \pi_2(a))(\sigma)$$

or

$$z \in \eta_X((\pi_2(a))(\sigma))$$

or

$$z \in \{y \mid y = (\pi_2(a))(\sigma)\}$$

or

$$z = (\pi_2(a))(\sigma),$$

giving us the final and positive result

$$a, \sigma, z \models \quad (\pi_2(a))(\sigma) = z \iff z = (\pi_2(a))(\sigma)$$

$$a \models \quad a = a$$

Distributitivty of Multiplication We consider,

$$\models \quad \rho_X \circ \mu_{FX} = F(\mu_X) \circ \rho_{\mathbf{P}X} \circ \mathbf{P}(\rho_X)$$

$$A \models \quad \rho_X(\mu_{FX}(A)) = F(\mu_X)(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A)))$$

Once again, considering both cases separately:

$$A \models \quad \rho_1(\mu_{FX}(A)) = \pi_1(F(\mu_X)(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A))))$$

$$A \models \quad \rho_1(\mu_{FX}(A)) = \pi_1(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A)))$$

$$A \models \quad \rho_1(\mu_{FX}(A)) = \rho_1(\mathbf{P}(\rho_X)(A))$$

$$A \models \quad \exists \langle \varepsilon, \delta \rangle \in \mu_{FX}(A).\, \varepsilon \iff \exists \langle \varepsilon, \delta \rangle \in \mathbf{P}(\rho_X)(A).\, \varepsilon$$

Here again, we consider the right and the left hand of the implication separately: On the left we can expand $\mu_{FX}(A)$,

$$\exists \langle \varepsilon, \delta \rangle . \langle \varepsilon, \delta \rangle \in \{x \mid \exists s.\, s \in A \wedge x \in s\} \wedge \varepsilon$$

or equivalently,

$$\exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \langle \varepsilon, \delta \rangle \in s) \wedge \varepsilon.$$

On the other side, we can expand $\mathbf{P}(\rho_X)$

$$\exists \langle \varepsilon, \delta \rangle \in \mathbf{P}(\rho_X)(A).\, \varepsilon$$

which going by $x \in P(f)(s) \iff \exists y.\, y \in s \wedge f(y) = x$, is

$$\exists \langle \varepsilon, \delta \rangle . \langle \varepsilon, \delta \rangle \in \{x \mid \exists y.\, y \in A \wedge \rho_X(y) = x\} \wedge \varepsilon,$$

or

$$\exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \rho_X(s) = \langle \varepsilon, \delta \rangle) \wedge \varepsilon,$$

or

$$\exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \rho_1(s) = \varepsilon \wedge \rho_2(s) = \delta) \wedge \varepsilon,$$

where we can disregard δ, as it doesn't interest us beyond its existence,

$$\exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge (\exists \langle \varepsilon', \delta' \rangle \in s.\, \varepsilon') = \varepsilon) \wedge \varepsilon,$$

which simplifies to

$$\exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \langle \varepsilon, \delta \rangle \in s) \wedge \varepsilon,$$

as δ' was not constrained. This gives us the intended equality result for ρ_1.

For the second case, we once again fix an arbitrary σ:

$$A, \sigma \models \quad \rho_2(\mu_{FX}(A))(\sigma) = \pi_2(F(\mu_X)(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A))))(\sigma)$$
$$A, \sigma \models \quad \rho_2(\mu_{FX}(A))(\sigma) = ({\mu_X}^{\Sigma}(\pi_2(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A)))))(\sigma)$$
$$A, \sigma \models \quad \rho_2(\mu_{FX}(A))(\sigma) = \mu_X(\pi_2((\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A))))(\sigma))$$
$$A, \sigma, x \models \quad x \in \rho_2(\mu_{FX}(A))(\sigma) \iff x \in \mu_X(\pi_2((\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A))))(\sigma))$$

Simplispanding the LHS of the implication we get

$$x \in \rho_2(\mu_{FX}(A))(\sigma)$$
$$\iff x \in \{y \mid \exists \langle \varepsilon, \delta \rangle \in \mu_{FX}(A).\, \delta(\sigma) = y\}$$
$$\iff x \in \{y \mid \exists \langle \varepsilon, \delta \rangle \in \{z \mid \exists s.\, s \in A \wedge z \in s\}.\, \delta(\sigma) = y\}$$
$$\iff x \in \{y \mid \exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \langle \varepsilon, \delta \rangle \in s).\, \delta(\sigma) = y\}$$
$$\iff \exists \langle \varepsilon, \delta \rangle . (\exists s.\, s \in A \wedge \langle \varepsilon, \delta \rangle \in s).\, \delta(\sigma) = x$$

The the RHS, consider

$$x \in \mu_X(\pi_2((\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A))))(\sigma))$$
$$\iff x \in \{y \mid \exists s.\, s \in \pi_2(\rho_{\mathbf{P}X}(\mathbf{P}(\rho_X)(A)))(\sigma) \wedge y \in s\}$$
$$\iff x \in \{y \mid \exists s.\, s \in (\rho_2(\mathbf{P}(\rho_X)(A)))(\sigma) \wedge y \in s\}$$
$$\iff x \in \{y \mid \exists s.\, s \in \{z \mid \exists \langle \varepsilon, \delta \rangle \in \mathbf{P}(\rho_X)(A).\, \delta(\sigma) = z\} \wedge y \in s\}$$
$$\iff x \in \{y \mid \exists s.\, (\exists \langle \varepsilon, \delta \rangle \in \mathbf{P}(\rho_X)(A).\, \delta(\sigma) = s) \wedge y \in s\}$$

where $\mathbf{P}(\rho_X)(A) = \{w \mid \exists s.\, v \in A \wedge w = \rho_X(v)\}$,

$$\iff x \in \{y \mid \exists s.\, (\exists \langle \varepsilon, \delta \rangle . (\exists v.\, v \in A \wedge \langle \varepsilon, \delta \rangle = \rho_X(v)) \wedge \delta(\sigma) = s) \wedge y \in s\}$$

allowing us once again to ignore ε,

$$\iff x \in \{y \mid \exists s.\, (\exists \langle \varepsilon, \delta \rangle \,.\, (\exists v.\, v \in A \land \delta = \rho_2(v)).\, \delta(\sigma) = s) \land y \in s\}$$

$$\iff \exists s.\, (\exists \langle \varepsilon, \delta \rangle \,.\, (\exists v.\, v \in A \land \delta = \rho_2(v)) \land \delta(\sigma) = s) \land x \in s$$

$$\iff \exists s.\, (\exists \langle \varepsilon, \delta \rangle \,.\, \exists v.\, v \in A \land \rho_2(v)(\sigma) = s) \land x \in s$$

$$\iff \exists \langle \varepsilon, \delta \rangle \,.\, \exists v.\, v \in A \land x \in \rho_2(v)(\sigma)$$

$$\iff \exists \langle \varepsilon, \delta \rangle \,.\, \exists v.\, v \in A \land x \in \rho_2(v)(\sigma)$$

keeping in mind that $\rho_2(v) = (\sigma \mapsto \{x \mid \exists \langle \varepsilon, \delta \rangle \in v \land \delta(\sigma) = x\})$,

$$\iff \exists \langle \varepsilon, \delta \rangle \,.\, \exists v.\, v \in A \land x \in \{x \mid \exists \langle \varepsilon, \delta \rangle \in v \land \delta(\sigma) = x\}$$

$$\iff \exists \langle \varepsilon, \delta \rangle \,.\, \exists v.\, v \in A \land \exists \langle \varepsilon', \delta' \rangle \in v \land \delta'(\sigma) = x$$

$$\iff \exists s.\, s \in A \land \exists \langle \varepsilon, \delta \rangle \in s \land \delta(\sigma) = x$$

This leaves us with the question,

$$\exists \langle \varepsilon, \delta \rangle \,.\, (\exists s \in A.\, \langle \varepsilon, \delta \rangle \in s).\, \delta(\sigma) = x \stackrel{?}{\iff} \exists s \in A.\, \exists \langle \varepsilon, \delta \rangle \in s.\, \delta(\sigma) = x$$

which holds by the commutativity of $\exists$ given that $\langle \varepsilon, \delta \rangle$ is "free" in A.

This concludes the proof, demonstrating that the $\mathcal{EM}$-distributive law for holds in $\mathscr{E}$ for ρ_X. ■