Some notes on the distributive law

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Given a monad $(T : \mathsf{Set} \to \mathsf{Set}, \eta, \mu)$, where $TX = \wp(X)$ and a Set endofunctor $FX = 1 + \Sigma \times X$, the distributive law[1] or $\mathcal{EM}$-law ("Eilenberg-Moore" law) should be the natural transformation

$$\rho : TF \Rightarrow FT,$$

or point-wise

$$\rho_X : \wp(1 + \Sigma \times X) \cong 2 \times \wp(\Sigma \times X) \cong 2 \times \wp(X)^{\Sigma} \to 1 + \Sigma \times \wp(X).$$

This "distributes" the monad T "under" a functor F.

First attempt First let us consider a possible definition of ρ_X,

$$\rho_X : 2 \times \wp(X)^{\Sigma} \to 1 + \Sigma \times \wp(X)$$

$$\rho_X(b, f) = \begin{cases} \iota_1\left(\boxed{1}, f\left(\boxed{2}\right)\right) & \text{if } b = \top \\ \iota_2 * & \text{if } b = \bot \end{cases}$$

where $\boxed{1}$ and $\boxed{2}$ represent two holes of the type Σ that cannot be satisfied given the context $b : 2, f : \wp(X)^{\Sigma}$ along with the state space[2].

Second attempt For the sake of completeness, consider the "converse" distributive law $\kappa : FT \Rightarrow TF$ (which is actually the $\mathcal{K}\ell$-law), that is once again point-wise as expected

$$\kappa_X : 1 + \Sigma \times \wp(X) \to 2 \times \wp(X)^{\Sigma}.$$

Here we at fist succeed given a intuitive enough definition

$$\kappa_X(x) = \begin{cases} (\bot, \varnothing) & \text{if } x = \iota_2 * \\ (\top, \boxed{\sigma'} \mapsto s) & \text{if } x = \iota_1\left(\boxed{\sigma}, s\right) \end{cases}$$

which turns out to be natural wrt. a $f : X \to Y$ and

$$\mathrm{id}_2 \times \wp(f)^{\Sigma} \circ \kappa_X = \kappa_Y \circ [\mathrm{id}_1; \mathrm{id}_{\Sigma} \times \wp(f)]$$

yet is not satisfying since it has two instances of unused values $\boxed{\sigma}$ and $\boxed{\sigma'}$.

Contextualisation What are ρ and κ expressing? Applying intuition from automata theory, we can attempt to convince ourselves if these transformations should be definable in the first place:

1. The type of ρ_X describes

   > *Given a nondeterministic automaton N over a state space X and an input alphabet Σ, indicate a final state $* \in 1$ **or** return a letter $\sigma \in \Sigma$ along with a set of states of X.*

   As seen above, the non-accepting states of N can be mapped onto the final state $*$. We could pick an arbitrary σ (going by the working-assumption that Σ is non-empty) and then use N to determine the successor state, but with no further information about σ this would seen non-intuitive (and non-constructive).

   This appears to confirm the issue encountered above.

2. The type of κ_X describes

[1] The composition represents a non-deterministic automaton with the input alphabet Σ.
[2] Unless we return $\iota_2 *$ in both cases, which would be natural but pointless

*Given a final state **or** a pair of a letter $\sigma \in \Sigma$ and a set of states of X, construct a nondeterministic automaton N, i.e. indicate if the current state is accepting or not and provide a map to transition from the current set of states to a new set of states, while possibly taking a $\sigma' \in \Sigma$ into account.*

The object in $1 + \Sigma \times \wp(X)$ can indicate failure or a successful transition *by a specific letter in* Σ. A new attempt at defining a κ' might attempt to make use of this interpretation:

$$\kappa'_X(x) = \begin{cases} (\bot, \varnothing) & \text{if } x = \iota_2 * \\ \left(\top, \sigma' \mapsto \begin{cases} \varnothing & \text{if } \sigma \neq \sigma' \\ s & \text{if } \sigma = \sigma' \end{cases}\right) & \text{if } x = \iota_1(\sigma, s) \end{cases}.$$

(Uncertain:) Coalgebraically FT are potentially infinite trees branching by subset of X, with letters of Σ as nodes. With κ' we translate these into NA, but the mapping is "injective" since we are reconstructing the inner workings from a input sequence $\langle \sigma_1 \sigma_2 \dots \rangle \in \Sigma^*$ along with a (tree-)tace. In this sense we appear to be non-determinising our input.

This remains unsatisfying, as $FT \Rightarrow TF$ is not what we are looking for in terms of a $\mathcal{EM}$distributive law. But using κ' we can consider if we could find an inverse (or why we cannot, which of course is not a proof that there exists no $FT \Rightarrow TF$).

A trivial inverse map κ'^{-1} where $\kappa'^{-1}\kappa'$ is an identity map can determine what $\sigma \in \Sigma$ N from the trace κ' uses to construct N and re-create the trace. It is easy to see that $\kappa'^{-1}\kappa'$ wouldn't work for an arbitrary NA, the assumption that only a single $\sigma \in \Sigma$ is accepted in a given state doesn't generally hold.

So the question is can we construct a ρ that will *naturally* construct a trace given an arbitrary NA? Intuitively it appears that we cannot do so, without making arbitrary choices, which confirms the initial impression, where ρ had two holes of type Σ.

Alternative Approach The existence of a $\mathcal{EM}$-law $\rho : TF \Rightarrow FT$ corresponds bijectively to the lifting of an endofunctor F on a category $\mathscr{C}$ to a endofunctor on $\mathcal{EM}(T)$:[3]

$$\begin{tikzcd} \mathcal{EM}(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{EM}(T) \arrow[d] \\ \mathscr{C} \arrow[r, "F"] & \mathscr{C} \end{tikzcd}$$

where $\mathscr{C} = \mathsf{Set}$ in our case.

So if given a $\hat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$, then we should be able to construct an adequate ρ.

Preliminary Conclusion It seems that the indended result does not arise obviously the combination of T and F. Assuming that there are no algebraic tricks that I have forgot to notice, together with the apparent lack of an intuitive and natural description of what $TF \Rightarrow FT$ designates, it appears that there is no solution.

Jacobs, et. al. give a further indication that the mistake might lie in the initial problem statement, when they develop "Non-deterministic Automata in $\mathcal{EM}$-style":[4]

[3]Bart Jacobs, Alexandra Silva, and Ana Sokolova. "Trace semantics via determinization". In: *International Workshop on Coalgebraic Methods in Computer Science.* Springer. 2012, pp. 109–129, p. 113.

[4]Ibid., p. 117.

[...] It yields a $\mathcal{EM}$-law with the components $\rho = \rho_1 \times \rho_2 : \wp\left(2 \times X^A\right) \to 2 \times \wp\left(X\right)^A$, given by:

$$\begin{cases} \rho_1(U) = 1 & \iff \exists h \in X^A.\, \langle 1, h\rangle \in U \\ x = \rho_2(U)(a) & \iff \exists \langle b, h\rangle \in U.\, h(a) = x \end{cases}$$

where in their case $A = \Sigma$, $TX = \wp\left(X\right)$ and $FX = 2 \times X^A$, **not** $1 + A \times X$. The intuition, that Jacobs' ρ maps a subset of deterministic automata into a single non-deterministic automaton (where ρ_1 says if any DA accepts a word, the NA will accept it as well) is easily imaginable.