An Introduction to Dependent Type Theory

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24.07.2025

Seminar Wissensrepräsentation und -verarbeitung

Roadmap

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In this talk we will:

- Extend $\lambda\!\rightarrow$ to first-order logic (resulting in $\lambda\mathrm{P}$)

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- Extend $\lambda\!\to$ to first-order logic (resulting in $\lambda \mathrm{P}$)
- Discuss properties of $\lambda \mathrm{P}$

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- Discuss properties of $\lambda\mathrm{P}$
- Look at applications of dependent type theory

Recall: Propositional Logic to FOL

Definition

The syntax of propositional logic is defined by:

$$\varphi, \psi ::= \bot \mid P \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \qquad P \in \mathcal{V}$$

where $\mathcal{V}$ is a set of propositional variables.

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First-order logic extends propositional logic with predicates and quantifiers:

$$\varphi, \psi ::= \bot \mid P(t_1, \ldots, t_n) \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \forall x.\varphi \mid \exists x.\varphi$$

where $t_1, \ldots, t_n$ are terms and P is a n-ary predicate.

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Extending $\lambda{\rightarrow}$ with Dependent Types

Recall: The System $\lambda{\rightarrow}$

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\begin{aligned}
t, s &::= x \mid \lambda(x : \varphi).t \mid t\ s && \text{(Terms)} \\
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Judgements of the form $\Gamma \vdash t : \varphi$ can be derived via:

$$
\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ (Ax)
$$

$$
\frac{\Gamma, x : \varphi \vdash t : \psi}{\Gamma \vdash \lambda(x : \varphi).t : \varphi \to \psi}\ (\to_i)
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- The type **Fin n** of numbers smaller than **n**.

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Remark

$\mathbf{Vec}\mathbb{N}$ and **Fin** themselves are **not** types, but type-families (indexed over natural numbers).

However, all of $\mathbf{Vec}\mathbb{N}$ **2**, **Fin 42**, $\mathbf{Vec}\mathbb{N}$ **123** are types.

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Judgements are of the form:

$\Gamma \vdash t : \varphi$ (Typing) $\qquad$ $\Gamma \vdash \varphi : \kappa$ (Kinding)

$\Gamma \vdash \kappa$ (Kind formation)

The System λP

Typing rules:

$$\frac{\Gamma \vdash \varphi : *}{\Gamma, x : \varphi \vdash x : \varphi}\ (Ax_t)$$

$$\frac{\Gamma, x : \varphi \vdash t : \psi}{\Gamma \vdash \lambda(x : \varphi).t : \forall(x : \varphi).\psi}\ (\forall_i) \qquad \frac{\Gamma \vdash t : \forall(x : \varphi).\psi \qquad \Gamma \vdash s : \varphi}{\Gamma \vdash t\ s : \psi[x := s]}\ (\forall_e)$$

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Kinding rules:

$$\frac{\Gamma \vdash \kappa}{\Gamma, X : \kappa \vdash X : \kappa}\ (Ax_\varphi)$$

$$\frac{\Gamma, x : \varphi \vdash \psi : \star}{\Gamma \vdash \forall(x : \varphi).\psi : \star}\ (\Pi_i) \qquad \frac{\Gamma \vdash \varphi : (\Pi x : \psi).\kappa \qquad \Gamma \vdash t : \psi}{\Gamma \vdash \varphi\ t : \kappa[x := t]}\ (\Pi_e)$$

Kind formation rules:

$$\frac{}{\Gamma \vdash *}\,(Ax_\kappa) \qquad \frac{\Gamma, x : \varphi \vdash \kappa}{\Gamma \vdash \Pi(x : \varphi).\kappa}\,(\kappa_i)$$

Example
Let us derive

$$\Gamma \vdash sum\ 4\ v : \mathbb{N}$$

with

$$\begin{aligned}\Gamma = \{&\mathbb{N} : \star,\\ &Vec\mathbb{N} : \mathbb{N} \Rightarrow \star,\\ &sum : \forall(n : \mathbb{N}).\, Vec\mathbb{N}\ n \to \mathbb{N},\\ &4 : \mathbb{N},\\ &v :\ Vec\mathbb{N}\ 4\}\end{aligned}$$

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$$\Gamma \vdash \forall(n : \mathbb{N}).\forall(m : \mathbb{N}).eq_{\mathbb{N}}\ (add\ n\ m)\ (add\ m\ n) : \star$$

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$$\begin{aligned}\Gamma = \{&\mathbb{N} : \star, \\ &add : \mathbb{N} \to \mathbb{N} \to \mathbb{N}, \\ &eq_{\mathbb{N}} : \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \star\}\end{aligned}$$

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Theorem

Let φ be a first-order formula consisting only of $\to$ and $\forall$. There exists a λ-term t such that $\Gamma_\Sigma \vdash t : \varphi$ iff φ is a theorem of intuitionistic FOL with signature Σ.

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Theorem

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Remark (for further reading)

The Curry-Howard isomorphism for "full" FOL requires us to extend $\lambda\mathrm{P}$ with the following constructs:

- Product, sum and empty type (corresponding to $\wedge, \vee, \bot$)
- Dependent sum type (corresponding to existential quantification)

Properties of λP

Expressiveness of $\lambda\mathrm{P}$

Let us define a translation from $\lambda\mathrm{P}$ to $\lambda{\to}$:

Terms

$$\overline{x} := x$$

$$\overline{t\ s} := \bar{t}\ \overline{s}$$

$$\overline{\lambda(x:\varphi).t} := \lambda(x:\overline{\varphi}).\bar{t}$$

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Corollary

$\lambda\mathrm{P}$ can express (i.e. assign a type to) **exactly** *the same terms as $\lambda{\to}$.*

More Properties

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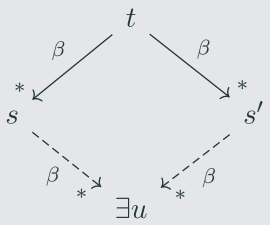

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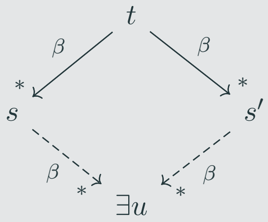

Corollary

For every term in λP *there* **exists** *a* **unique** *normal form.*

Applications of dependent types

The Curry-Howard Isomorphism in Practice

(Un)Safe Programming

Consider the following Haskell code:

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data ListN where
  Nil  :: ListN
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This could benefit from dependent types!

Safe Programming

In Agda this can be expressed in a **safe** and **correct** way:

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data VecN : ℕ → * where
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We can use Agda to do first-order proofs (and more!):

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+-assoc : ∀ (m : ℕ) (n : ℕ) (o : ℕ)
          → ((m + n) + o) ≡ (m + (n + o))
+-assoc zero     n o = refl
+-assoc (succ m) n o = cong succ (+-assoc m n o)
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