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ERLANGEN-NÜRNBERG

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CHAIR FOR COMPUTER SCIENCE 8
THEORETICAL COMPUTER SCIENCE

Nominal Coalgebraic Modal Logic

Master Thesis in Computer Science

Leon Vatthauer

Advisor:

Lutz Schröder

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Erlangen, 5. Oktober 2026 _____
Leon Vatthauer

Abstract

TODO

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1 Introduction

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2 Preliminaries

2.1 Group Theory

Let us recall some basic definitions of group theory:

Definition 2.1.1 (Group). A *group* $(G, \cdot)$ is a set G together with a binary composition operator $\cdot : G \times G \rightarrow G$ such that

1. there exists an $e \in G$ such that $x \cdot e = x = e \cdot x$ for every $x \in G$,
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for every $x, y, z \in G$,
3. for every $x \in G$ there exists an x^{-1} such that $x \cdot x^{-1} = e = x^{-1} \cdot x$.

Definition 2.1.2 (Group Homomorphism). A *group homomorphism* $f : (G, \cdot) \rightarrow (H, \times)$ is a mapping $f : G \rightarrow H$ such that

$$f(x \cdot y) = f(x) \times f(y),$$

for all $x, y \in G$. Oftentimes we will use the same composition operator for both groups, since one can infer from the context which one is used at which position.

Example 2.1.3. The theory of nominal sets is based on the group of finite permutations:

$$\mathfrak{S}_f(A) := \{\pi : A \rightarrow A \mid \pi \text{ bijective and finite}\},$$

where π being finite means that its support

$$\text{supp}(\pi) := \{a \in A \mid \pi(a) \neq a\}$$

is finite.

Definition 2.1.4 (Group Action). The *action* of a group $(G, \cdot)$ on a set X , also called *G-set*, is a binary operator $\cdot : G \times X \rightarrow X$ such that

1. $e \cdot x = x$,
2. $(g \cdot h) \cdot x = g \cdot (h \cdot x)$,

for $g, h \in G$ and $x \in X$. Note that we use the same notation for the group composition and the group action, this is motivated by the fact that every group can be viewed as a group action of the group itself on its underlying set.

Example 2.1.5. For every set A we can define a $\mathfrak{S}_f(A)$ -Set with the action

$$\pi \cdot a = \pi(a),$$

for $\pi \in \mathfrak{S}_f(A), a \in A$.

Definition 2.1.6 (Equivariance). A relation $R \subseteq X \times Y$ on G -sets $(X, \cdot)$ and $(Y, \cdot)$ is called *equivariant* if it is closed under the group action i.e.,

$$xRy \Rightarrow (g \cdot x)R(g \cdot y),$$

for $g \in G, x \in X, y \in Y$.

For functions $f : X \rightarrow Y$ this condition simplifies to

$$f(g \cdot x) = g \cdot f(x),$$

for $g \in G, x \in X$.

2.2 Nominal Sets

For the rest of this thesis we fix a countably infinite set $\mathbb{A} = \{a, b, \dots\}$ of variable names.

Definition 2.2.1 (Support). Let X be a $\mathfrak{S}_f(\mathbb{A})$ -set, we call a set $S \subseteq \mathbb{A}$ a *support* of an element $x \in X$, if for all $\pi \in \mathfrak{S}_f(\mathbb{A})$:

$$(\forall a \in S. \pi(a) = a) \Rightarrow \pi \cdot x = x$$

Definition 2.2.2 (Nominal Set). A *nominal set* is a $\mathfrak{S}_f(\mathbb{A})$ -set, where every element has a finite support.

Lemma 2.2.3 (Pitts [9]). *Finite supports are closed under intersection.*

Definition 2.2.4 (Least support). Let $(X, \cdot)$ be a nominal set and $x \in X$. Since every x has a finite support, the intersection of its supports must also be finite, thus we define the *least support* as

$$\begin{aligned} \text{supp}_X(x) &= \bigcap \{S \subseteq \mathbb{A} \mid (\forall a \in S. \pi(a) = a) \Rightarrow \pi \cdot x = x\} \\ &= \{a \in \mathbb{A} \mid a \text{ is in every finite support of } x\} \end{aligned}$$

which indeed supports x by Lemma 2.2.3.

Definition 2.2.5 (Freshness). $a \in \text{supp}(x)$, means x depends on the variable a , conversely, when $a \notin \text{supp}(x)$, we call a *fresh* for x and write $a \# x$.

Definition 2.2.6 (Orbits and Subsets). The *orbit* of $x \in X$ is the set of elements reachable by permutation, i.e., $\text{orb}(x) := \{\pi \cdot x \mid \pi \in \mathfrak{S}_f(\mathbb{A})\}$.

A nominal set is *orbit-finite* if it is a union of finitely many orbits. For a nominal set X , a subset of X is called an *orbit-finite subset* if it is finitely supported and contained in a finite union of orbits of X .

Lemma 2.2.7. *Let $f : X \rightarrow Y$ be an equivariant function. Then the following hold:*

1. *For $x \in X$, $f[\text{orb}(x)] = \text{orb}(f(x))$, i.e., equivariant functions preserve orbits;*
2. *and if $U \subseteq X$ is an orbit-finite subset of X , then $f[U]$ is an orbit-finite subset of Y .*

Proof. The first claim follows easily by equivariance of f :

$$f[\text{orb}(x)] = \{f(\pi \cdot x) \mid \pi \in \mathfrak{S}_f(\mathbb{A})\} = \{\pi \cdot f(x) \mid \pi \in \mathfrak{S}_f(\mathbb{A})\} = \text{orb}(f(x)).$$

For the second claim we need to verify two properties, by equivariance of f and Lemma 2.2.11, $f[U]$ is (finitely) supported by $\text{supp}(U)$. Thus, we are left to show that $f[U]$ is contained in a finite union of orbits of Y . Note that U is contained in a finite union of orbits of X , meaning $U \in X_1 \cup \dots \cup X_n$, for a finite number of orbits of X . Thus, $f[U] \in f[X_1] \cup \dots \cup f[X_n]$ and by the first claim, each X_i is an orbit. $\square$

Definition 2.2.8 (Orbit-finite Hull). Let X be a nominal set. The (*orbit-finite*) *hull* of a finite subset $Y \subseteq X$ under a finite set of atoms A is defined as

$$\text{hull}_A(Y) := \{\pi \cdot y \mid y \in Y, \pi \in \mathfrak{S}_f(\mathbb{A}), \forall a \in A. \pi(a) = a\}.$$

$\text{hull}_A(Y)$ is generally not an equivariant subset of X , we write $\text{hull}(Y) := \text{hull}_\emptyset(Y)$ for the equivariant orbit-finite hull of X . For infinite subsets $Y \subseteq X$, $\text{hull}(Y)$ denotes the equivariant closure of Y .

Note that $\text{hull}_A(Y) \in \mathcal{P}_{\text{fs}}(X)$, since $\text{hull}_A(Y)$ is supported by A . Furthermore, it is even orbit-finite as expected by the construction. What is less obvious is that every orbit-finite subset can be represented as an orbit-finite hull.

Lemma 2.2.9 ([9, Pitts, Proposition 5.25]). *Let X be a nominal set.*

1. *For each finite $A \subseteq \mathbb{A}$ and finite $Y \subseteq X$, $\text{hull}_A(Y) \in \mathcal{P}_{\text{of}}(X)$.*
2. *Every $S \in \mathcal{P}_{\text{of}}(X)$ is equal to $\text{hull}_{\text{supp}(S)}(Y)$ for some finite $Y \subseteq X$.*

Example 2.2.10 (Constructing Nominal Sets). Let X, Y be nominal sets, constructions on nominal sets (with their least supports) include:

1. The product $X \times Y$ is defined as on sets with

$$\pi \cdot (x, y) = (\pi \cdot x, \pi \cdot y)$$

and

$$\text{supp}_{X \times Y}(x, y) = \text{supp}_X(x) \cup \text{supp}_Y(y).$$

2. The *fresh product* $X * Y$ is a subset of $X \times Y$ given by

$$X * Y = \{(x, y) \in X \times Y \mid x \# y\}.$$

Permutations and support behave the same way as for the regular product.

3. The coproduct or disjoint union $X + Y$ also defined as on sets with

$$\pi \cdot \text{inl}(x) = \text{inl}(\pi \cdot x)$$

$$\pi \cdot \text{inr}(y) = \text{inr}(\pi \cdot y)$$

and

$$\text{supp}_{X+Y}(\text{inl}(x)) = \text{supp}_X(x)$$

$$\text{supp}_{X+Y}(\text{inr}(y)) = \text{supp}_Y(y)$$

4. The finitely supported powerset

$$\mathcal{P}_{\text{fs}}(X) := \{M \in \mathcal{P}(X) \mid M \text{ has finite support}\}$$

with

$$\pi \cdot M = \{\pi \cdot x \mid x \in M\}$$

5. The orbit finite powerset

$$\mathcal{P}_{\text{of}}(X) := \{M \in \mathcal{P}_{\text{fs}}(X) \mid M \text{ orbit-finite subset of } X\}$$

6. The finitely supported function space

$$X \rightarrow_{\text{fs}} Y := \{f : X \rightarrow Y \mid f \text{ has finite support}\}$$

with

$$\pi \cdot f(x) := \pi \cdot f(\pi^{-1} \cdot x)$$

and

$$\text{supp}_Y(f(x)) \subseteq \text{supp}_{(X \rightarrow_{\text{fs}} Y)}(f) \cup \text{supp}_X(x).$$

Sometimes we write Y^X for $X \rightarrow_{\text{fs}} Y$, especially when considering the function that arises from this construction.

7. Given an equivariant equivalence relation $\approx : X \times X$, the quotient of X under $\approx$ is the quotient set $X/\approx$ with

$$\pi \cdot [x]_{\approx} := [\pi \cdot x]_{\approx}$$

and

$$\text{supp}_{X/\approx}([x]_{\approx}) = \bigcap_{y \in [x]_{\approx}} \text{supp}_X(y),$$

where $[x]_{\approx}$ is the $\approx$ -equivalence class of $x \in X$.

lem:equivari
antsubset

Lemma 2.2.11. *For every equivariant function $f : X \rightarrow Y$ and $x \in X$,*

$$\text{supp}(f(x)) \subseteq \text{supp}(x).$$

Proof. Since $\text{supp}(f(x))$ is the least support of $f(x)$, it suffices to show that $\text{supp}(x)$ is a support of $f(x)$. Thus let π be a permutation that fixes x , then

$$\pi \cdot f(x) = f(\pi \cdot x) = f(x).$$

□

Definition 2.2.12 (α -equivalence). Let X be a nominal set. α -equivalence is an equivariant equivalence relation $=_{\alpha} \subseteq (\mathbb{A} \times X) \times (\mathbb{A} \times X)$ given by

$$(a, x) =_{\alpha} (b, y) : \Longleftrightarrow \exists c \in \mathbb{A}, c \notin \text{supp}(a, b, x, y) \wedge (a \ c) \cdot x = (b \ c) \cdot y.$$

Definition 2.2.13 (Abstraction Set). We write $[\mathbb{A}]X$ for the quotient $(\mathbb{A} \times X)/=_{\alpha}$ called the *abstraction set* (later: *abstraction functor*). Equivalence classes $(a, x) \in \mathbb{A} \times X$ are denoted by $\langle a \rangle x$.

When considering the abstraction set for finitely supported powersets, the following definition will be helpful:

Definition 2.2.14 (α -equivalent-subset). For two equivalence classes $\langle a \rangle S, \langle b \rangle T \in [\mathbb{A}]\mathcal{P}_{\text{fs}}X$, we define the equivariant α -equivalent-subset-relation $\subseteq_{\alpha} \subseteq [\mathbb{A}]\mathcal{P}_{\text{fs}}X \times [\mathbb{A}]\mathcal{P}_{\text{fs}}X$ by

$$\langle a \rangle S \subseteq_{\alpha} \langle b \rangle T : \Longleftrightarrow \exists c. c \notin \text{supp}(a, b, S, T) \wedge (a \ c) \cdot S \subseteq (b \ c) \cdot T.$$

2.3 Category Theory

Category theory is a tool for abstracting many mathematical notions into structures called categories:

Definition 2.3.1 (Category). A *category* $\mathcal{C}$ consists of

- a collection of objects $|\mathcal{C}|$,
- for every two objects $C, D \in |\mathcal{C}|$ a set of morphisms $\text{hom}(C, D)$ such that
- for every $A \in |\mathcal{C}|$ there is a morphism $(\text{id} : A \rightarrow A) \in \text{hom}(A, A)$,
- there is a composition operator on morphisms written $((f : B \rightarrow C) \circ (g : A \rightarrow B)) : A \rightarrow C$ for $A, B, C \in |\mathcal{C}|$, such that
- for every $f : A \rightarrow B$, $f \circ \text{id} = f = \text{id} \circ f$ and
- $h \circ (g \circ f) = (h \circ g) \circ f$ for $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$.

In this thesis we will restrict ourselves to the category Nom of nominal sets, let us verify that this is indeed a category:

Example 2.3.2 (Category Nom). The category Nom consists of nominal sets as objects and equivariant functions as morphisms, this indeed constitutes a category because:

- the identity function on sets is equivariant: $\text{id}(\pi \cdot x) = \pi \cdot x = \pi \cdot \text{id}(x)$,
- composition is composition on set functions, which preserves equivariance: Given $f : B \rightarrow C, g : A \rightarrow B$ that are both equivariant, then $f(g(\pi \cdot x)) = f(\pi \cdot g(x)) = \pi \cdot f(g(x))$,
- and the axioms of identity and composition are inherited from the category Set of sets.

Let us now recapitulate some basic concepts of category theory but in the category of nominal sets. For a general treatment of category theory refer to [6].

Definition 2.3.3 (Functor). A *functor* on Nom consists of

- a mapping from every $C \in |\text{Nom}|$ to an object $FC \in |\text{Nom}|$,
- a mapping from every equivariant function $f : C \rightarrow D$ in Nom to an equivariant function $Ff : FC \rightarrow FD$ in Nom .
- Such that $F(\text{id}) = \text{id}$ and
- $F(f \circ g) = Ff \circ Fg$.

Note that functors on a single category are usually called endofunctors, however since we operate only within the category Nom , we will refer to them simply as functors.

Example 2.3.4. Let us provide some exemplary functors by giving their object mappings and function mappings:

1. Id is defined by

$$\begin{aligned} \text{Id}(X) &= X \\ \text{Id}(f : X \rightarrow Y) &= f : X \rightarrow Y. \end{aligned}$$

2. $F \circ G$ is the composition of two functors, defined by

$$\begin{aligned} (F \circ G)(X) &= F(G(X)) \\ (F \circ G)(f : X \rightarrow Y) &= F(G(f)) : F(G(X)) \rightarrow F(G(Y)). \end{aligned}$$

ex:functors

3. $F^{\text{op}} : \text{Nom}^{\text{op}} \rightarrow \text{Nom}^{\text{op}}$ is the opposite functor for a functor F on Nom , where the opposite category Nom^{op} has the same objects as Nom , but morphisms are reversed. It is defined by

$$\begin{aligned} F^{\text{op}}X &= FX \\ F^{\text{op}}(f : X \rightarrow Y) &= Ff, \end{aligned}$$

so it differs from F only by its signature, since Nom and Nom^{op} are not equal.

4. The product of two functors $F \times G$ is defined by

$$\begin{aligned} (F \times G)(X) &= FX \times GX \\ (F \times G)(f : X \rightarrow Y) &= Ff \times Gf. \end{aligned}$$

5. $\mathcal{P}_{\text{fs}}$ extends to a functor called the finitely supported covariant powerset functor with

$$\mathcal{P}_{\text{fs}}(f : X \rightarrow Y) = (S \in \mathcal{P}_{\text{fs}}(X)) \mapsto f[S].$$

6. $\mathcal{P}_{\text{of}}$ extends to a functor called the orbit-finite powerset functor with

$$\mathcal{P}_{\text{of}}(f : X \rightarrow Y) = (S \in \mathcal{P}_{\text{fs}}(X)) \mapsto f[S].$$

7. $\mathcal{Q}_{\text{fs}} : \text{Nom}^{\text{op}} \rightarrow \text{Nom}$ is the finitely supported contravariant powerset functor defined by

$$\begin{aligned} \mathcal{Q}_{\text{fs}}(X) &= \mathcal{P}_{\text{fs}}(X) \\ \mathcal{Q}_{\text{fs}}(f : X \rightarrow Y) &= (S \in \mathcal{P}_{\text{fs}}(Y)) \mapsto f^{-1}[S]. \end{aligned}$$

8. $(-)^N$ is the exponentiation functor for any given nominal set N . It is defined by:

$$\begin{aligned} X^N &= \{f : N \rightarrow_{\text{fs}} X\} \\ (f : X \rightarrow Y)^N &= (g \in X^N) \mapsto f \circ g. \end{aligned}$$

9. The abstraction set $[\mathbb{A}]X$ extends to a functor called the *abstraction functor* defined by

$$[\mathbb{A}](f : X \rightarrow Y)(\langle a \rangle x) = \langle a \rangle f(x).$$

10. The fresh product $\mathbb{A} * X$ extends to a functor with

$$(\mathbb{A} * X)(f : X \rightarrow Y)(a, x) = (a, f(x)),$$

where $a \# f(x)$ due to equivariance of f .

Definition 2.3.5 (Adjoint Functors). Two functors L, R on Nom are *adjoint*, or part of an *adjunction*, if there exists a natural isomorphism

$$\text{Nom}(LX, Y) \cong \text{Nom}(X, RY).$$

We write $L \dashv R$ and say L is the *left-adjoint* and R is the *right-adjoint*.

Lemma 2.3.6 ([9, Theorem 4.12, Theorem 4.13]). *The abstraction functor has both a left and a right-adjoint, i.e., there exist functors L, R on Nom such that*

$$LX \dashv [\mathbb{A}]X \dashv RX.$$

These functors are given by

$$LX = \mathbb{A} * X = \{(a, x) \in \mathbb{A} \times X \mid a \# x\}$$

and

$$RX = \{f : \mathbb{A} \rightarrow_{\text{fs}} X \mid \forall a \in \mathbb{A}. a \# f(a)\}.$$

We call $\mathbb{A} * X$ the *freshness functor* and R , since it can be viewed to model deallocation, the *deallocation functor*. Since both are adjoint to the abstraction functor, the following interesting fact follows, which will be useful when thinking about logics with freshness or deallocation.

`lem:abstr-adj` **Lemma 2.3.7.** *Under the finitely supported powerset we have*

$$\mathcal{P}_{\text{fs}}(\mathbb{A} * X) \cong R(\mathcal{P}_{\text{fs}}X).$$

Proof. This uses several facts, first note that the abstraction functor preserves exponentials [9, Proposition 4.14] and $[\mathbb{A}]2 \cong 2$, since $2 = \{\top, \perp\}$ is discrete. Thus, the abstraction functor also preserves powersets, i.e.,

$$[\mathbb{A}](\mathcal{P}_{\text{fs}}X) \cong [\mathbb{A}]2^X \cong 2^{[\mathbb{A}]X} \cong \mathcal{P}_{\text{fs}}([\mathbb{A}]X). \quad (1) \text{ itm:cong-1}$$

Furthermore, on hom-sets the following isomorphisms hold:

$$\text{Nom}(X, \mathcal{P}_{\text{fs}}Y) \cong \text{Nom}(X, 2^Y) \cong \text{Nom}(X \times Y, 2) \cong \text{Nom}(Y, 2^X) \cong \text{Nom}(Y, \mathcal{P}_{\text{fs}}X). \quad (2) \text{ itm:cong-2}$$

Now, using the Yoneda lemma:

$$\begin{aligned} & \text{Nom}(Y, \mathcal{P}_{\text{fs}}(\mathbb{A} * X)) \\ & \cong \text{Nom}(\mathbb{A} * X, \mathcal{P}_{\text{fs}}Y) \quad (2) \\ & \cong \text{Nom}(X, [\mathbb{A}](\mathcal{P}_{\text{fs}}Y)) \quad (\text{Lemma 2.3.7}) \\ & \cong \text{Nom}(X, \mathcal{P}_{\text{fs}}([\mathbb{A}]Y)) \quad (1) \\ & \cong \text{Nom}([\mathbb{A}]Y, \mathcal{P}_{\text{fs}}X) \quad (2) \\ & \cong \text{Nom}(Y, R(\mathcal{P}_{\text{fs}}X)) \quad (\text{Lemma 2.3.7}) \quad \square \end{aligned}$$

Definition 2.3.8 (Algebra and Coalgebra). Let F be a functor on Nom . An F -algebra is a tuple $(C, \alpha : FC \rightarrow C)$, where C is a nominal set and α an equivariant function.

Dually, an F -coalgebra is a tuple $(C, \beta : C \rightarrow FC)$, where again C is a nominal set and β an equivariant function.

Definition 2.3.9 (Coalgebra Homomorphism). Let F be a functor on Nom . A *homomorphism between two F -coalgebras* (A, α) and (B, β) is an equivariant function $f : A \rightarrow B$, such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \beta \\ FA & \xrightarrow{Ff} & FB \end{array}$$

`def:coalgsum` **Definition 2.3.10** (Sum of Coalgebras). The sum of two coalgebras $((C, \gamma) + (D, \delta))$ is defined as the sum $C + D$ of the underlying sets together with the equivariant map $\xi : C + D \rightarrow F(C + D)$ given by

$$\begin{aligned} \xi(\text{inl}(c)) &= \gamma(c) \\ \xi(\text{inr}(d)) &= \delta(d) \end{aligned}$$

Note that $\gamma(c) \in FC$ and $\delta(d) \in FD$, to amend that, we require F to preserve injective equivariant functions (also called *monics*), in which case $FC, FD \subseteq F(C + D)$.

Thus, when F preserves monics, (C, γ) and (D, δ) are subcoalgebras of $(C + D, \xi)$ in the sense that $C, D \subseteq C + D$ and

$$\begin{aligned}\xi(\text{inl}(c)) &\in F(C + D) \\ \xi(\text{inr}(d)) &\in F(C + D)\end{aligned}$$

for all $c \in C, d \in D$.

As we have seen, taking the sum of two coalgebras requires the respective functors to preserve monics, luckily all functors that we will consider do preserve monics:

Lemma 2.3.11. *All functors in Example 2.3.4 preserve monics, given that the functors they are made of (if any) also preserve monics.*

Proof. Most functors in Example 2.3.4 act on functions exactly as their Set-counterparts, thus they preserve monics, since in Set every functor can be adjusted to preserve monics [3, Proof of Theorem 3.2]. Only the abstraction functor has no Set-counterpart, thus let us verify that it preserves monics:

Let $f : X \rightarrow Y$ be a monic in Nom and let $[\mathbb{A}](f)(\langle a \rangle x) = [\mathbb{A}](f)(\langle b \rangle y)$ for $\langle a \rangle x, \langle b \rangle y \in [\mathbb{A}]X$. Taking a fresh $c \in \mathbb{A}$ we can simplify this to

$$(a \ c) \cdot f(x) = (b \ c) \cdot f(y),$$

which by equivariance of f corresponds to

$$f((a \ c) \cdot x) = f((b \ c) \cdot y).$$

Now, by injectivity of f we have $(a \ c) \cdot x = (b \ c) \cdot y$, i.e., $\langle a \rangle x = \langle b \rangle y$ which was to be proven. $\square$

Definition 2.3.12 (Natural Transformation). A *natural transformation* is a mapping between two functors F, G . Concretely, a natural transformation consists of a family of equivariant functions

$$(\eta_X : FX \rightarrow GX)_{X \in |\text{Nom}|}$$

such that

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

commutes for every equivariant $f : X \rightarrow Y$.

3 Nominal Modal Logic

In this chapter we will ‘nominalize’ various modal logics from the literature, see [5] for an overview and coalgebraic treatment of these logics on Set. We will introduce all of these logics on trivial nominal sets such that they represent exactly the same logics as on Set and then add labels and binding to these logics. Note that we will focus on the modal operators, thus we consider only minimal logics with negation, conjunction and modalities. Other logical connectives can then be defined in the usual way and propositional atoms will later be introduced as nullary modalities.

3.1 Modal Logics

We will start off by recalling basic relational modal logic with Kripke frames.

Example 3.1.1 (Relational Modal Logic). Relational modal logic includes two modalities $\Box$ and $\Diamond$, where depending on context the operators have different meanings, we will stick with treating $\Box$ as “necessarily” and $\Diamond$ as “possibly”. The minimal logic containing these operators is then generated by

$$\varphi, \psi ::= \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics of relational modal logic is usually established using *Kripke frames*, which in our nominal setting consist of

- a nominal set S of states and
- an equivariant transition relation $R \subseteq S \times S$.

Then we define satisfaction of a formula at a state c by

$$\begin{aligned} c \models \neg\varphi &\iff c \not\models \varphi \\ c \models \varphi \wedge \psi &\iff c \models \varphi \text{ and } c \models \psi \\ c \models \Box\varphi &\iff \text{for all } c' \in C, cRc' \Rightarrow c' \models \varphi \\ c \models \Diamond\varphi &\iff \text{there is } c' \in C, cRc' \wedge c' \models \varphi \end{aligned}$$

Example 3.1.2 (Neighborhood Semantics).

TODO: Neighborhood

Example 3.1.3 (Graded Modal Logic).

TODO: Graded

Example 3.1.4 (Probabilistic Modal Logic).

TODO: Probabilistic

Example 3.1.5 (Conditional Logic).

TODO: Conditional

3.2 Labels, Binding and Deallocation

TODO: Talk about labeled transition systems

TODO: Introduce Binding Hennessy Milner Logic

TODO: Show how other logics would look like with labels/binding

TODO: Talk about L and R

4 Nominal Coalgebraic Modal Logic

TODO: Improve text

There is extensive research on the coalgebraic structure of modal logics [5] in $\mathbf{Set}$, in this chapter we will consider the coalgebraic structure of the modal logics introduced in Chapter 3. We start out by lifting the usual structures of coalgebraic semantics to $\mathbf{Nom}$, and then consider how each logic fits into this setting.

4.1 Initial Algebras in $\mathbf{Nom}$

TODO: Complete rewrite, how to structure this?

Recall from Chapter 3 that in nominal modal logic, behavior of modal operators might be binding or non-binding. In this chapter we will work on a level generality that unifies both types of behaviors, by working with signature functors for each modal operator. The resulting set of modal formulae is then the initial algebra for a functor modelling the connectives of formulae.

Let us first study the requirements for such an initial algebra to exist. In [1] Adámek et al. identify conditions for functors to have initial algebras.

First, note that $\mathbf{Nom}$ is a locally finitely presentable (lfp) category. Thus, by [1, Theorem 11.2.18] every accessible functor on $\mathbf{Nom}$ has an initial algebra. We do not need to concern us with the notion of accessible functor, for us, it suffices that finitary functors are accessible [1, Remark 11.2.15].

4.2 Coalgebraic Semantics

Definition 4.2.1 (Modal Signature). A *modal signature* is a discrete nominal set Λ of modal operators $\heartsuit$, where each modal operator comes with a finitary arity functor

$$\mathbf{Ar}_{\heartsuit} : \mathbf{Nom} \rightarrow \mathbf{Nom}.$$

Definition 4.2.2 (Modal Formulae). Given a modal signature Λ , the nominal set of *modal formulae (over Λ)* $\mathcal{F}(\Lambda)$ is inductively generated by the rules:

- If $\varphi \in \mathcal{F}(\Lambda)$, then also $\neg\varphi \in \mathcal{F}(\Lambda)$;
- for every orbit-finite subset of formulae $\Phi \in \mathcal{P}_{\text{of}}(\mathcal{F}(\Lambda))$, we have $\bigwedge \Phi \in \mathcal{F}(\Lambda)$;
- and if $t \in \mathbf{Ar}_{\heartsuit}(\mathcal{F}(\Lambda))$ for some $\heartsuit \in \Lambda$, then $\heartsuit(t) \in \mathcal{F}(\Lambda)$.

Additionally, we define the usual connectives via

- For $\Phi \in \mathcal{P}_{\text{of}}(\mathcal{F}(\Lambda))$ we define disjunction as $\bigvee \Phi := \neg \bigwedge \{\neg\varphi \mid \varphi \in \Phi\}$;
- truth and falsity are defined as $\top := \bigwedge \emptyset$ and $\perp := \bigvee \emptyset$;

`def:modform`

- (bi-) implication is defined using the binary instances of $\bigwedge$ and $\bigvee$ with negation.

Note that this construction of $\mathcal{F}(\Lambda)$ is valid in Nom , since it is the description of the initial algebra of

$$GX = X + \mathcal{P}_{\text{of}}(X) + \coprod_{\heartsuit \in \Lambda} \text{Ar}_{\heartsuit}(X).$$

Note that G is a finitary functor since its component functors are finitary and Nom is a lfp category, thus the initial algebra of G exists by the standard construction of Adámek [1].

Example 4.2.3. Let us instantiate Hennessy-Milner to this setting, the modal operators $[a]$ for $a \in \mathbb{A}$ now share the same behavior, which is abstracted in the functor $\text{Ar}_{[-]}$. Concretely, we have only one modality $[-] \in \Lambda$ with the arity functor $\text{Ar}_{[-]}(X) = \mathbb{A} \times X$. Thus, the formula $[a]\varphi$ formally is written as $[-](a, \varphi)$ for $(a, \varphi) \in \mathbb{A} \times \mathcal{F}(\Lambda)$, we will however usually stick to the former notation to denote the latter.

Now, the semantics of modalities can be modeled using predicate liftings:

Definition 4.2.4 (Predicate Lifting). For a functor F on Nom and a modal operator $\heartsuit \in \Lambda$ of modal operators, a *predicate lifting* is a natural transformation

$$\llbracket \heartsuit \rrbracket : \text{Ar}_{\heartsuit} \circ \mathcal{Q}_{\text{fs}} \rightarrow \mathcal{Q}_{\text{fs}} \circ F^{\text{op}}.$$

Naturality corresponds to commutation of the following diagram:

$$\begin{array}{ccc} \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(X)) & \xrightarrow{\llbracket \heartsuit \rrbracket_X} & \mathcal{Q}_{\text{fs}}(FX) \\ \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(f)) \uparrow & & \uparrow \mathcal{Q}_{\text{fs}}(Ff) \\ \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(Y)) & \xrightarrow{\llbracket \heartsuit \rrbracket_Y} & \mathcal{Q}_{\text{fs}}(FY) \end{array}$$

for every equivariant $f : X \rightarrow Y$.

Definition 4.2.5 (Semantics of NCL). The *extension* or *truth set* of a modal formula φ is the set of all states that satisfy φ .

Concretely, the extension is the unique equivariant function $\llbracket - \rrbracket : \mathcal{F}(\Lambda) \rightarrow \mathcal{Q}_{\text{fs}}(X)$, satisfying

1. $\llbracket \neg \varphi \rrbracket_X = X \setminus \llbracket \varphi \rrbracket_X$
2. $\llbracket \bigwedge \Phi \rrbracket_X = \bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket_X$
3. $\llbracket \heartsuit t \rrbracket_X = \gamma^{-1}[\llbracket \heartsuit \rrbracket_X(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_X)(t))]$.

Such a function indeed exists by the fact that $\mathcal{F}(\Lambda)$ is an initial algebra and all constructions in the definition of $\llbracket - \rrbracket$ are equivariant.

Example 4.2.6. Instantiating the logic to Hennessy-Milner logic once again, yields in the modal case

$$\llbracket [-](a, \varphi) \rrbracket_X = \gamma^{-1}[\llbracket [-] \rrbracket_X(a, \llbracket \varphi \rrbracket_X)].$$

Using the notation of Example 4.2.3, and adjusting the predicate liftings accordingly, this statement can be rewritten as

$$\llbracket [a](\varphi) \rrbracket_X = \gamma^{-1}[\llbracket [a] \rrbracket_X(\llbracket \varphi \rrbracket_X)],$$

which is exactly the expected condition.

Lemma 4.2.7. *The support of the extension $\llbracket \varphi \rrbracket \in \mathcal{Q}_{\text{fs}}(S)$ is bounded by*

$$\text{supp}(\llbracket \varphi \rrbracket) \subseteq \text{supp}(\varphi)$$

Proof. Follows directly by Lemma 2.2.11 and equivariance of $\llbracket - \rrbracket$. $\square$

Lemma 4.2.8. *The extension yields an equivariant satisfaction relation defined by*

$$s \models \varphi : \Longleftrightarrow s \in \llbracket \varphi \rrbracket,$$

which can equivalently be described by

$$\begin{aligned} s \models \neg \varphi &\Longleftrightarrow s \not\models \varphi \\ s \models \bigwedge \Phi &\Longleftrightarrow \forall \varphi \in \Phi. s \models \varphi \\ s \models \heartsuit t &\Longleftrightarrow \gamma(s) \in \llbracket \heartsuit \rrbracket_S(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket)(t)). \end{aligned}$$

Proof. The correspondence of both definitions is immediate by pattern matching the generators of $\mathcal{F}(\Lambda)$ and unfolding the definitions, equivariance follows from equivariance of the extension:

$$s \models \varphi \Longleftrightarrow s \in \llbracket \varphi \rrbracket \Longleftrightarrow \pi \cdot s \in \pi \cdot \llbracket \varphi \rrbracket \Longleftrightarrow \pi \cdot s \in \llbracket \pi \cdot \varphi \rrbracket \Longleftrightarrow \pi \cdot s \models \pi \cdot \varphi. \quad \square$$

Lemma 4.2.9 (Invariance under coalgebra morphisms). *Let (S, γ) and (T, δ) be nominal F -coalgebras and let $f : S \rightarrow T$ be an equivariant F -coalgebra morphism. Then*

$$\llbracket \varphi \rrbracket_S = f^{-1}[\llbracket \varphi \rrbracket_T]$$

for every $\varphi \in \mathcal{F}(\Lambda)$.

By Lemma 4.2.8 this lemma equivalently states that

$$f(s) \models \varphi \Longleftrightarrow s \models \varphi$$

for every $s \in S$ and $\varphi \in \mathcal{F}(\Lambda)$.

Proof. The condition can be stated in a pointfree manner as $\llbracket - \rrbracket_S = \mathcal{Q}_{\text{fs}}(f) \circ \llbracket - \rrbracket_T$, which follows from uniqueness of $\llbracket - \rrbracket_S$ as stated in Definition 4.2.5. Let us verify that $\mathcal{Q}_{\text{fs}}(f) \circ \llbracket - \rrbracket_T$ satisfies the same conditions as $\llbracket - \rrbracket_S$:

1. $f^{-1}[\llbracket \neg \varphi \rrbracket_T] = f^{-1}[T \setminus \llbracket \varphi \rrbracket_T] = S \setminus f^{-1}[\llbracket \varphi \rrbracket_T]$
2. $f^{-1}[\llbracket \bigwedge \Phi \rrbracket_T] = f^{-1}[\bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket_T] = \bigcap_{\varphi \in \Phi} f^{-1}[\llbracket \varphi \rrbracket_T]$
3. $f^{-1}[\llbracket \heartsuit(t) \rrbracket_T]$

$$\begin{aligned} &= f^{-1}[\delta^{-1}[\llbracket \heartsuit \rrbracket_T(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_T)(t))]] \\ &= (\delta \circ f)^{-1}[\llbracket \heartsuit \rrbracket_T(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_T)(t))] \\ &= (Ff \circ \gamma)^{-1}[\llbracket \heartsuit \rrbracket_T(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_T)(t))] && (f \text{ coalgebra morphism}) \\ &= \gamma^{-1}[(Ff)^{-1}[\llbracket \heartsuit \rrbracket_T(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_T)(t))]] \\ &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_S(\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(f))(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_T)(t)))] && (\text{naturalness}) \\ &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_S(\text{Ar}_{\heartsuit}(f^{-1}[\llbracket - \rrbracket_T])(t))] && (\text{functoriality of } \text{Ar}_{\heartsuit}) \end{aligned}$$

$\square$

We will now use this invariance result to follow one direction of a Hennesy-Milner theorem.

def:indistinguishability

Definition 4.2.10 (Logical Indistinguishability). Let $c \in C$ and $d \in D$ be states in two F -coalgebras, we call c, d $\mathcal{F}(\Lambda)$ -indistinguishable, if they satisfy the same formulae in $\mathcal{F}(\Lambda)$, i.e., if

$$C, c \models \varphi \iff D, d \models \varphi$$

for all $\varphi \in \mathcal{F}(\Lambda)$.

Since the satisfaction relation is equivariant, we can lift $\mathcal{F}(\Lambda)$ -indistinguishability to an equivariant relation

$$c \equiv_{\mathcal{F}(\Lambda)} d : \iff c \text{ and } d \text{ are } \mathcal{F}(\Lambda)\text{-indistinguishable.}$$

Definition 4.2.11 (Behavioral Equivalence). Let $c \in (C, \gamma)$ and $d \in (D, \delta)$ be states in two F -coalgebras, we call c, d *behaviorally equivalent* if there exists a third coalgebra (E, ϵ) and two unifying F -coalgebra morphisms $f : (C, \gamma) \rightarrow (E, \epsilon)$ and $g : (D, \delta) \rightarrow (E, \epsilon)$ with $f(c) = g(d)$.

In this case we write $c \sim_F d$.

TODO: Notes on bisimilarity.

cor:bisim-to

Corollary 4.2.12 (Lemma 4.2.9). *Behaviorally equivalent states are logically indistinguishable, i.e.,*

$$c \sim_F d \implies c \equiv_{\mathcal{F}(\Lambda)(\Gamma)} d.$$

4.3 Nominal Modal Logics Coalgebraically

Hennessy-Milner Logic

def:nomhml

The definition of nominal transition systems can be rewritten into a coalgebraic structure by encoding the equivariant relation as an equivariant function

$$S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}},$$

thus yielding the coalgebra $(S, \gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}})$, where $\mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}$ is the application of the composition of the functors $(-)^{\mathbb{A}}$ and $\mathcal{P}_{\text{fs}}$ as defined in Example 2.3.4. Concretely, the functor action is given by:

$$\begin{aligned} (S \rightarrow T) &\rightarrow (\mathcal{P}_{\text{fs}}(S)^{\mathbb{A}} \rightarrow \mathcal{P}_{\text{fs}}(T)^{\mathbb{A}}) \\ (f : S \rightarrow T)(g : \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}})(a : \mathbb{A}) &\mapsto f[g(a)]. \end{aligned}$$

Lemma 4.3.1. *Let us verify the correspondence of equivariant relations $R \subseteq S \times \mathbb{A} \times S$ and equivariant functions $S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}$.*

Proof.

$\implies$ Let $R \subseteq S \times \mathbb{A} \times S$ be an equivariant relation.

We define the function $\gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}$ by

$$\gamma(s)(a) = \{s' \in S \mid sR(a, s')\}$$

and prove its equivariance:

$$\begin{aligned}
& (\pi \cdot \gamma(s))(a) \\
&= \pi \cdot \gamma(s)(\pi^{-1} \cdot a) \\
&= \{\pi \cdot s' \in S \mid sR((\pi^{-1} \cdot a), s')\} \\
&= \{s' \in S \mid sR(\pi^{-1} \cdot a, \pi^{-1} \cdot s')\} \\
&= \{s' \in S \mid (\pi \cdot s)R(a, s')\} \\
&= \gamma(\pi \cdot s)(a).
\end{aligned}$$

For correctness, we are left to show that $\gamma(s)$ has finite support. Indeed, $\text{supp}(s)$ also supports $\gamma(s)$, since by equivariance we have:

$$\pi \cdot \gamma(s) = \gamma(\pi \cdot s) = \gamma(s),$$

for any π that fixes all elements of $\text{supp}(s)$.

$\Leftarrow$ Let $\gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}$ be an equivariant function.

We define $R \subseteq S \times \mathbb{A} \times S$ as the relation

$$R = \{(s, a, s') \mid s \in S, a \in \mathbb{A}, s' \in \gamma(s)(a)\}.$$

This relation is equivariant: let $sR(a, s')$, i.e., $s \in S, a \in \mathbb{A}$ and $s' \in \gamma(s)(a)$. We need to show that $(\pi \cdot s)R(\pi \cdot a, \pi \cdot s')$, i.e., we are left to show that $(\pi \cdot s') \in \gamma(\pi \cdot s)(\pi \cdot a)$, which holds, by equivariance of γ :

$$\gamma(\pi \cdot s)(\pi \cdot a) = ((\pi \cdot \gamma)(s))(\pi \cdot a) = \pi \cdot \gamma(s)(\pi^{-1} \cdot \pi \cdot a) = \pi \cdot \gamma(s)(a),$$

and by assumption, we have $(\pi \cdot s') \in (\pi \cdot \gamma(s)(a))$. □

The predicate liftings for nominal transition systems $(S, \gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}})$ are given by

$$\begin{aligned}
\llbracket [-] \rrbracket : \mathbb{A} \times \mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}) \\
(a, B) &\mapsto \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(S) \mid f(a) \subseteq B\}.
\end{aligned}$$

Of course, by currying the arguments we can view the predicate lifting as a function

$$\begin{aligned}
\llbracket [a] \rrbracket : \mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{P}_{\text{fs}}(S)^{\mathbb{A}}) \\
B &\mapsto \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(S) \mid f(a) \subseteq B\}
\end{aligned}$$

for $a \in \mathbb{A}$.

We are left to show that this construction is equivariant and natural, where equivariance follows by the equivariance principle. For naturality, consider an equivariant map $f : X \rightarrow Y$, we need to show that $\llbracket [a] \rrbracket_X \circ \mathcal{Q}_{\text{fs}}(f) = \mathcal{Q}_{\text{fs}}(\mathcal{P}_{\text{fs}}(f)^{\mathbb{A}}) \circ \llbracket [a] \rrbracket_Y$ for all $a \in \mathbb{A}$, which amounts to

$$h \in \llbracket [a] \rrbracket_X(f^{-1}[Z]) \iff \mathcal{P}_{\text{fs}}(f)^{\mathbb{A}}(h) \in \llbracket [a] \rrbracket_Y(Z),$$

for $Z \in \mathcal{P}_{\text{fs}}(Y)$ and $h \in \mathcal{P}_{\text{fs}}(X)^{\mathbb{A}}$. This can further be unfolded to

$$h(a) \subseteq f^{-1}[Z] \iff f[h(a)] \subseteq Z,$$

which trivially holds.

Neighborhood Logics

TODO: Explain neighborhood semantics

TODO: Add restriction to monotone as subfunctor

Consider the *finitely supported neighborhood functor* $\mathcal{N}_{\text{fs}}$ defined by

$$\begin{aligned}\mathcal{N}_{\text{fs}}(X) &:= \mathcal{P}_{\text{fs}}(\mathcal{P}_{\text{fs}}(X)) \\ \mathcal{N}_{\text{fs}}(f : X \rightarrow Y)(N) &:= \{Z \in \mathcal{P}_{\text{fs}}(Y) \mid f^{-1}[Z] \in N\}\end{aligned}$$

for $X, Y \in |\text{Nom}|$, $f : X \rightarrow Y$ equivariant and $N \in \mathcal{N}_{\text{fs}}(X)$, and let there be a coalgebra

$$\gamma : S \rightarrow \mathcal{N}_{\text{fs}}(S)^{\mathbb{A}}.$$

We define predicate liftings via:

$$\begin{aligned}\llbracket [a] \rrbracket_S &: \mathcal{Q}_{\text{fs}}(S) \rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{N}_{\text{fs}}(S)^{\mathbb{A}}) \\ B &\mapsto \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(S) \mid B \in f(a)\}.\end{aligned}$$

Again, we need to verify equivariance and naturality, starting off with equivariance:

$$\begin{aligned}\pi \cdot \llbracket [a] \rrbracket_S(B) &= \{\pi \cdot f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(S) \mid B \in f(a)\} \\ &= \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(S) \mid B \in (\pi^{-1} \cdot f)(a)\} \\ &= \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(S) \mid B \in \pi^{-1} \cdot f(\pi \cdot a)\} \\ &= \{f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(S) \mid \pi \cdot B \in f(\pi \cdot a)\} \\ &= \llbracket [\pi \cdot a] \rrbracket_S(\pi \cdot B)\end{aligned}$$

For naturality, we have to verify that

$$(\mathcal{N}_{\text{fs}}(f)^{\mathbb{A}})(g) \in \llbracket [a] \rrbracket_Y(Z) \iff g \in \llbracket [a] \rrbracket_X(f^{-1}[Z]),$$

holds for $f : X \rightarrow Y$, $g : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{N}_{\text{fs}}(X)$ and $Z \in \mathcal{P}_{\text{fs}}(X)$. The condition can be simplified to

$$Z \in (\mathcal{N}_{\text{fs}}(f)^{\mathbb{A}})(g)(a) \iff f^{-1}[Z] \in g(a),$$

which holds, since

$$\begin{aligned}Z &\in (\mathcal{N}_{\text{fs}}(f)^{\mathbb{A}})(g)(a) \\ \iff Z &\in \mathcal{N}_{\text{fs}}(f)(g(a)) \\ \iff f^{-1}[Z] &\in g(a).\end{aligned}$$

Graded Nominal Modal Logic

Bag functor:

$$\begin{aligned}\mathcal{B}_{\text{fs}}(X) &= X \rightarrow_{\text{fs}} \mathbb{N}_{\infty} \\ \mathcal{B}_{\text{fs}}(f : X \rightarrow Y)(\mu : X \rightarrow_{\text{fs}} \mathbb{N}_{\infty})(y) &= \sum_{x \in f^{-1}(y)} \mu(x)\end{aligned}$$

Coalgebras:

$$S \rightarrow \mathcal{B}_{\text{fs}}(S)^{\mathbb{A}}.$$

Predicate Lifting:

$$\begin{aligned} \llbracket \langle a \rangle_{>k} \rrbracket_S &: \mathcal{Q}_{\text{fs}}(S) \rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{B}_{\text{fs}}(S)^{\mathbb{A}}) \\ Y &\mapsto \{ \mu : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{B}_{\text{fs}}(S) \mid \sum_{y \in Y} \mu(a)(y) > k \} \end{aligned}$$

Equivariance:

$$\begin{aligned} & \pi \cdot \llbracket \langle a \rangle_{>k} \rrbracket_S(Y) \\ &= \{ \pi \cdot \mu \mid \sum_{y \in Y} \mu(a)(y) > k \} \\ &= \{ \mu \mid \sum_{y \in Y} (\pi^{-1} \cdot \mu)(a)(y) > k \} \\ &= \{ \mu \mid \sum_{y \in Y} (\pi^{-1} \cdot \mu(\pi \cdot a))(y) > k \} \\ &= \{ \mu \mid \sum_{y \in Y} \pi^{-1} \cdot \mu(\pi \cdot a)(\pi \cdot y) > k \} \\ &= \{ \mu \mid \sum_{y \in Y} \mu(\pi \cdot a)(\pi \cdot y) > k \} \\ &= \{ \mu \mid \sum_{\pi^{-1} \cdot y \in Y} \mu(\pi \cdot a)(y) > k \} \\ &= \{ \mu \mid \sum_{y \in \pi \cdot Y} \mu(\pi \cdot a)(y) > k \} \\ &= \llbracket \pi \cdot \langle a \rangle_{>k} \rrbracket_S(\pi \cdot Y) \end{aligned}$$

Naturality: Let $f : S \rightarrow T$ be an equivariant map. We need to verify $\llbracket \langle a \rangle_{>k} \rrbracket_S \circ \mathcal{Q}_{\text{fs}}(f) = \mathcal{Q}_{\text{fs}}(\mathcal{B}_{\text{fs}}(f)^{\mathbb{A}}) \circ \llbracket \langle a \rangle_{>k} \rrbracket_T$, which amounts to

$$\mu \in \llbracket \langle a \rangle_{>k} \rrbracket_S(f^{-1}[Z]) \iff (\mathcal{B}_{\text{fs}}(f)^{\mathbb{A}})(\mu) \in \llbracket \langle a \rangle_{>k} \rrbracket_T(Z)$$

for $\mu \in \mathcal{B}_{\text{fs}}(S)^{\mathbb{A}}$ and $U \in \mathcal{P}_{\text{fs}}(T)$.

This can be verified by unfolding the definitions:

$$\sum_{s \in f^{-1}[U]} \mu(a)(s) > k \iff \sum_{t \in U} \sum_{s \in f^{-1}(t)} \mu(a)(s) > k \iff \sum_{t \in U} (\mathcal{B}_{\text{fs}}(f)^{\mathbb{A}})(\mu)(a)(t) > k.$$

Probabilistic Nominal Modal Logic

Discrete probability functor:

$$\mathcal{D}_{\text{fs}}(X) = \{ \mu : X \rightarrow_{\text{fs}} [0, 1] \mid \sum_{x \in X} \mu(x) = 1 \}$$

$$\mathcal{D}_{\text{fs}}(f : X \rightarrow Y)(\mu : X \rightarrow_{\text{fs}} [0, 1])(y) = \sum_{x \in f^{-1}(y)} \mu(x)$$

Where

$$\sum_{y \in Y} \mathcal{D}_{\text{fs}}(f)(\mu)(y) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \mu(x) = \sum_{x \in f^{-1}[Y]} \mu(x) = 1.$$

Coalgebras:

$$C \rightarrow \mathcal{D}_{\text{fs}}(C)^{\mathbb{A}}$$

Predicate Lifting:

$$\begin{aligned} \llbracket \langle a \rangle_{\geq p} \rrbracket : \mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{D}_{\text{fs}}(S)^{\mathbb{A}}) \\ Y &\mapsto \{ \mu : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{D}_{\text{fs}}(S) \mid \sum_{y \in Y} \mu(a)(y) \geq p \} \end{aligned}$$

Equivariance and naturality like in the graded case.

Conditional Logic

Functor

$$\begin{aligned} \mathcal{C}_{\text{fs}}(X) &= \mathcal{Q}_{\text{fs}}(X) \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(X) \\ \mathcal{C}_{\text{fs}}(f : X \rightarrow Y)(g : \mathcal{Q}_{\text{fs}}(X) \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(X))(Z) &= f[g(f^{-1}[Z])] \end{aligned}$$

TODO: Restriction to $\mathcal{C}_{ID}$ as subfunctor.

Coalgebras

$$C \rightarrow \mathcal{C}_{\text{fs}}(C)^{\mathbb{A}}$$

TODO: Somewhat related to [12].

Predicate Lifting:

$$\begin{aligned} \llbracket \Rightarrow_a \rrbracket : \mathcal{Q}_{\text{fs}}(S) \times \mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{C}_{\text{fs}}(S)^{\mathbb{A}}) \\ (X, Y) &\mapsto \{ f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}(S) \mid f(a)(X) \subseteq Y \} \end{aligned}$$

Equivariance:

$$\begin{aligned} &\pi \cdot \llbracket \Rightarrow_a \rrbracket(X, Y) \\ &= \{ \pi \cdot f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}(S) \mid f(a)(X) \subseteq Y \} \\ &= \{ f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}(S) \mid (\pi^{-1} \cdot f)(a)(X) \subseteq Y \} \\ &= \{ f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}(S) \mid \pi^{-1} \cdot f(\pi \cdot a)(\pi \cdot X) \subseteq Y \} \\ &= \{ f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}(S) \mid f(\pi \cdot a)(\pi \cdot X) \subseteq \pi \cdot Y \} \\ &= \llbracket \pi \cdot \Rightarrow_a \rrbracket(\pi \cdot X, \pi \cdot Y) \end{aligned}$$

Naturality: Let $f : X \rightarrow Y$ equivariant. We need to verify

$$\llbracket \Rightarrow_a \rrbracket_X \circ (\mathcal{Q}_{\text{fs}}(f) \times \mathcal{Q}_{\text{fs}}(f)) = \mathcal{Q}_{\text{fs}}(\mathcal{C}_{\text{fs}}(f)^{\mathbb{A}}) \circ \llbracket \Rightarrow_a \rrbracket_Y.$$

For $g : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{C}_{\text{fs}}(X)$ and $f : X \rightarrow Y$ and $Z, U \subseteq Y$, it suffices to show

$$g \in \llbracket \Rightarrow_a \rrbracket_X(f^{-1}[Z], f^{-1}[U]) \iff (\mathcal{C}_{\text{fs}}(f))^{\mathbb{A}}(g) \in \llbracket \Rightarrow_a \rrbracket_Y(Z, U),$$

which follows by

$$\begin{aligned} & g \in \llbracket \Rightarrow_a \rrbracket_X(f^{-1}[Z], f^{-1}[U]) \\ \iff & g(a)(f^{-1}[Z]) \subseteq f^{-1}[U] \\ \iff & f[g(a)(f^{-1}[Z])] \subseteq U \\ \iff & (\mathcal{C}_{\text{fs}}(f))^{\mathbb{A}}(g)(a)(Z) \subseteq U \\ \iff & (\mathcal{C}_{\text{fs}}(f))^{\mathbb{A}}(g) \in \llbracket \Rightarrow_a \rrbracket_Y(Z, U). \end{aligned}$$

Propositional Atoms

Let F be a Nom-functor, with a coalgebra $\gamma : S \rightarrow FS$ and a modal signature Λ . These elements describe the semantics of a logic $\mathcal{L}$. We can add *propositional atoms* to this logic, by treating them like nullary modal operators.

Formally, consider a nominal set At . First, we extend the modal signature by setting $\Lambda'_0 := \Lambda_0 \uplus \text{At}$ and adjusting the corresponding predicate liftings accordingly. Now, since propositional atoms are either true or false in a given state, their semantics can be captured in an equivariant function

$$\delta : S \rightarrow \mathcal{P}_{\text{fs}}(\text{At})$$

where each state is mapped to the set of atoms that it satisfies.

Using this we extend the aforementioned semantic structures by setting

$$GS = FS \times \mathcal{P}_{\text{fs}}(\text{At})$$

on objects and

$$Gf = Ff \times \text{id}$$

on functions $f : S \rightarrow T$ and furthermore extending the coalgebra to $\langle \gamma, \delta \rangle : S \rightarrow FS \times \mathcal{P}_{\text{fs}}(\text{At})$

Finally, we define predicate liftings for each $p \in \text{At}$:

$$\llbracket p \rrbracket_S = \{(T, U) \mid p \in U\} \in \mathcal{P}_{\text{fs}}(FS \times \mathcal{P}_{\text{fs}}(\text{At}))$$

which are indeed equivariant:

$$\begin{aligned} & \pi \cdot \llbracket p \rrbracket_S \\ = & \{(\pi \cdot Y, \pi \cdot U) \mid p \in U\} \\ = & \{(Y, U) \mid p \in U\} \\ = & \{(Y, U) \mid \pi \cdot p \in U\} \\ = & \llbracket \pi \cdot p \rrbracket_S \end{aligned}$$

and satisfy naturality, i.e., the condition that

$$(Ff \times \text{id})(s, A) \in \llbracket p \rrbracket_T \iff (s, A) \in \llbracket p \rrbracket_S,$$

for $f : S \rightarrow T, s \in FS, A \in \mathcal{P}_{\text{fs}}(\text{At})$, since both sides of the equivalence simplify to $p \in A$.

Binding Hennessy-Milner Logic

Definition 4.3.2 (Binding Transition System). *Binding transition systems* feature labeled states as well as binding states, this is incorporated in the coalgebra:

$$\gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}} \times [\mathbb{A}]\mathcal{P}_{\text{fs}}(S).$$

By (REF FUSION) it suffices to introduce a predicate lifting for $[a]$ and fuse it with the liftings of nominal Hennessy-Milner logic Section 4.3.

$$\begin{aligned} \llbracket [-] \rrbracket : [\mathbb{A}]\mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}([\mathbb{A}]\mathcal{P}_{\text{fs}}(S)) \\ Q &\mapsto \{R \in [\mathbb{A}]\mathcal{P}_{\text{fs}}(S) \mid \forall a. R@a \subseteq Q@a\} \end{aligned}$$

$$\begin{aligned} \langle [-] \rangle : [\mathbb{A}]\mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}([\mathbb{A}]\mathcal{P}_{\text{fs}}(S)) \\ Q &\mapsto \{R \in [\mathbb{A}]\mathcal{P}_{\text{fs}}(S) \mid \forall a. R@a \cap Q@a \neq \emptyset\} \end{aligned}$$

Equivariance follows by equivariance of concretion and the freshness quantifier. For naturality, let $f : X \rightarrow Y$ be an equivariant map. We need to show that $\llbracket [-] \rrbracket_X \circ [\mathbb{A}]\mathcal{Q}_{\text{fs}}(f) = \mathcal{Q}_{\text{fs}}([\mathbb{A}]\mathcal{P}_{\text{fs}}(f)) \circ \llbracket [-] \rrbracket_Y$, which amounts to

$$p \in \llbracket [-] \rrbracket_X([\mathbb{A}]\mathcal{Q}_{\text{fs}}(f)(q)) \iff [\mathbb{A}]\mathcal{P}_{\text{fs}}(f)(p) \in \llbracket [-] \rrbracket_Y(q).$$

for $p \in [\mathbb{A}]\mathcal{P}_{\text{fs}}X, q \in [\mathbb{A}]\mathcal{P}_{\text{fs}}Y$.

Choose an $a \# p, q$, then we have

$$([\mathbb{A}]\mathcal{Q}_{\text{fs}}(f)(q))@a = f^{-1}[q@a] \quad \text{and} \quad ([\mathbb{A}]\mathcal{P}_{\text{fs}}(f)(p))@a = f[p@a].$$

Thus, it suffices to establish

$$p@a \subseteq f^{-1}[q@a] \iff f[p@a] \subseteq q@a,$$

which of course holds.

TODO: Mention fusion, merge propositional atoms and binding underneath?

4.4 A Hennessy-Milner Logic with Deallocation

TODO: Cite [10].

Definition 4.4.1 (Deallocating Transition System).

$$\gamma : S \rightarrow \mathcal{P}_{\text{fs}}(S)^{\mathbb{A}} \times [\mathbb{A}]\mathcal{P}_{\text{fs}}(S) \times R(\mathcal{P}_{\text{fs}}(S)) \times \mathcal{P}_{\text{fs}}(S).$$

$$\begin{aligned} \llbracket [-] \rrbracket : \mathbb{A} * \mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(R(\mathcal{P}_{\text{fs}}(S))) \\ (a, U) &\mapsto \{f \in R(\mathcal{P}_{\text{fs}}(S)) \mid f(a) \subseteq U\} \end{aligned}$$

TODO: Define $[\mathbb{A}*] = (\mathbb{A} * X)/\approx_\alpha$, talk about identity iso.

$$\begin{aligned} \llbracket [-] \rrbracket : [\mathbb{A}*]\mathcal{Q}_{\text{fs}}(S) &\rightarrow \mathcal{Q}_{\text{fs}}(\mathcal{P}_{\text{fs}}(S)) \\ \langle a \rangle U &\mapsto \{H \in \mathcal{P}_{\text{fs}}(S) \mid H \subseteq U\} \end{aligned}$$

5 Expressiveness of NML

5.1 Separation and Expressiveness

Definition 5.1.1 (Separation). Given a nominal set X , we call a modal signature Λ *separating at X* if every $t \in FX$ is uniquely determined by the set

$$\{(\heartsuit, Y) \mid \heartsuit \in \Lambda, Y \in \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(X)), t \in \llbracket \heartsuit \rrbracket(Y)\} \subseteq \Lambda \times \prod_{\heartsuit \in \Lambda} \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(X)).$$

A modal signature Λ is *separating*, if Λ is separating at every set X , in other words, if for every nominal set X the equivariant map

$$(t \in FX) \mapsto \{(\heartsuit, Y) \mid \heartsuit \in \Lambda, Y \in \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(X)), t \in \llbracket \heartsuit \rrbracket(Y)\},$$

is injective.

Lemma 5.1.2. *The map in Definition 5.1.1 is equivariant.*

Proof. Using equivariance of $\llbracket \heartsuit \rrbracket$ and discreteness of Λ (i.e., $\pi \cdot \heartsuit = \heartsuit$ for every $\heartsuit \in \Lambda$),

$$\begin{aligned} & \pi \cdot \{(\heartsuit, Y) \mid \heartsuit \in \Lambda, Y \in \heartsuit(\mathcal{P}_{\text{fs}}(X)), t \in \llbracket \heartsuit \rrbracket(Y)\} \\ &= \{(\heartsuit, \pi \cdot Y) \mid \heartsuit \in \Lambda, Y \in \heartsuit(\mathcal{P}_{\text{fs}}(X)), t \in \llbracket \heartsuit \rrbracket(Y)\} \\ &= \{(\heartsuit, Y) \mid \heartsuit \in \Lambda, Y \in \heartsuit(\mathcal{P}_{\text{fs}}(X)), t \in \llbracket \heartsuit \rrbracket(\pi^{-1} \cdot Y)\} \\ &= \{(\heartsuit, Y) \mid \heartsuit \in \Lambda, Y \in \heartsuit(\mathcal{P}_{\text{fs}}(X)), \pi \cdot t \in \llbracket \heartsuit \rrbracket(Y)\}. \end{aligned}$$

□

Example 5.1.3 (Separation). Separation holds in our running examples:

1. *Nominal Modal Logic:* Let $f, g \in \mathcal{P}_{\text{fs}}(X)^{\mathbb{A}}$ such that for all $(a, Y) \in \mathbb{A} \times \mathcal{P}_{\text{fs}}(X)$: $f \in \llbracket [a] \rrbracket(Y)$ iff $g \in \llbracket [a] \rrbracket(Y)$. We have to show that $f = g$, so let $a \in \mathbb{A}$, then we have $f \in \llbracket [a] \rrbracket(f(a))$, which implies $g \in \llbracket [a] \rrbracket(f(a))$, i.e., $g(a) \subseteq f(a)$. The other inclusion is then shown symmetrically, thus $f = g$.
2. *Neighborhood logic:* Let $N, M \in \mathcal{N}_{\text{fs}}(X)$ such that for all $a \in \mathbb{A}$ and $Y \in \mathcal{P}_{\text{fs}}(X)$: $N \in \llbracket [a] \rrbracket(Y)$ iff $M \in \llbracket [a] \rrbracket(Y)$. By the definition of $\llbracket [a] \rrbracket$, this yields that $Y \in N$ iff $Y \in M$, thus $N = M$ as required.
3. *Binding Nominal Logic:* Let $\langle b \rangle S, \langle c \rangle T \in [\mathbb{A}] \mathcal{P}_{\text{fs}}(X)$ such that for all $Q \in [\mathbb{A}] \mathcal{P}_{\text{fs}}(X)$:

$$\langle b \rangle S \in \llbracket [-] \rrbracket(Q) \iff \langle c \rangle T \in \llbracket [-] \rrbracket(Q). \quad (\text{ii}) \quad \text{eq:t2}$$

We need to verify that $\langle b \rangle S = \langle c \rangle T$, which holds iff $T = (b \ c) \cdot S$. Let us proceed using antisymmetry of the subset relation. By instantiating $Q := \langle b \rangle S$ in Equation (ii), we get:

$$\exists B. \langle b \rangle S = \langle b \rangle B \wedge S \subseteq B \iff \exists B. \langle b \rangle S = \langle c \rangle B \wedge T \subseteq B.$$

The left existential statement holds by setting $B := S$, yielding

$$\exists B. \langle b \rangle S = \langle c \rangle B \wedge T \subseteq B,$$

which in turn yields a B such that $B := (b \ c) \cdot S$ and thus $T \subseteq (b \ c) \cdot S$.

Finally, the second inclusion follows by a symmetrical argument where we put $Q := \langle c \rangle T$ and $B := (b \ c) \cdot T$ in the right existential statement.

4. *Graded Nominal Modal Logic:* Let $\mu, \nu \in \mathcal{B}_{\text{fs}}(S)^{\mathbb{A}}$ such that for all $a \in \mathbb{A}, k \in \mathbb{N}$ and $U \in \mathcal{P}_{\text{fs}}(S)$: $\mu \in \llbracket \langle a \rangle_{>k} \rrbracket_S(U) \iff \nu \in \llbracket \langle a \rangle_{>k} \rrbracket_S(U)$.

By definition of $\llbracket \langle a \rangle_{>k} \rrbracket_S$ this yields that

$$\sum_{s \in U} \mu(a)(s) > k \iff \sum_{s \in U} \nu(a)(s) > k,$$

for every $a \in \mathbb{A}, k \in \mathbb{N}, U \in \mathcal{P}_{\text{fs}}(S)$. Let us consider three cases:

- a) $\sum_{s \in U} \mu(a)(s) = 0$: then $\sum_{s \in U} \mu(a)(s) > k$ holds for no $k \in \mathbb{N}$, thus $\sum_{s \in U} \nu(a)(s) > k$ also never holds, which implies that also $\sum_{s \in U} \nu(a)(s) = 0$.
 - b) $\sum_{s \in U} \mu(a)(s) = \infty$: then $\sum_{s \in U} \mu(a)(s) > k$ holds for every $k \in \mathbb{N}$, and so does $\sum_{s \in U} \nu(a)(s) > k$, thus also $\sum_{s \in U} \nu(a)(s) = \infty$.
 - c) Otherwise: instantiate $k := \sum_{s \in U} \mu(a)(s) - 1$ and $k := \sum_{s \in U} \nu(a)(s) - 1$ respectively, yielding the desired equality by antisymmetry.
5. *Probabilistic Nominal Modal Logic:* Let $\mu, \nu \in \mathcal{D}_{\text{fs}}(X)^{\mathbb{A}}$ such that for all $a \in \mathbb{A}, p \in \mathbb{Q} \cap [0, 1]$ and $Y \in \mathcal{P}_{\text{fs}}(X)$: $\mu \in \llbracket \langle a \rangle_{\geq p} \rrbracket(Y) \iff \nu \in \llbracket \langle a \rangle_{\geq p} \rrbracket(Y)$.

By definition of $\llbracket \langle a \rangle_{\geq p} \rrbracket$ this yields that

$$\sum_{y \in Y} \mu(a)(y) \geq p \iff \sum_{y \in Y} \nu(a)(y) \geq p,$$

for every $a \in \mathbb{A}, p \in \mathbb{Q} \cap [0, 1], Y \in \mathcal{P}_{\text{fs}}(X)$. Let us prove equality by contradiction, assuming w.l.o.g. that

$$\sum_{y \in Y} \mu(a)(y) < \sum_{y \in Y} \nu(a)(y),$$

which implies that there is a $p \in \mathbb{Q} \cap [0, 1]$ such that (w.l.o.g.)

$$\sum_{y \in Y} \mu(a)(y) < p \leq \sum_{y \in Y} \nu(a)(y).$$

But by the assumption this also implies that $\sum_{y \in Y} \mu(a)(y) \geq p$, a contradiction. Thus, $\mu = \nu$.

6. *Conditional Logic:* Let $f, g : \mathcal{C}_{\text{fs}}(X)^{\mathbb{A}}$ such that for all $a \in \mathbb{A}$ and $U, V \in \mathcal{P}_{\text{fs}}(X)$: $f \in \llbracket \Rightarrow_a \rrbracket(U, V) \iff g \in \llbracket \Rightarrow_a \rrbracket(U, V)$. By the definition of $\llbracket \Rightarrow_a \rrbracket$, this yields that

$$f(a)(U) \subseteq V \iff g(a)(U) \subseteq V,$$

by instantiating $V := f(a)(U)$ and $V := g(a)(U)$ respectively, we obtain the required pointwise equality $f(a)(U) = g(a)(U)$.

Definition 5.1.4 (Expressiveness). Let F be a functor on Nom and Λ a modal signature. We call $\mathcal{F}(\Lambda)$ *expressive* if for any two F -coalgebras $(X, \gamma), (Y, \delta)$, and for $x \in X, y \in Y$ the following holds:

$$x \equiv_{\mathcal{F}(\Lambda)} y \rightarrow x \sim_F y.$$

def:
orbitbounded

Definition 5.1.5 (Orbit-finitely bounded functor). A functor $F : \text{Nom} \rightarrow \text{Nom}$ is *orbit-finitely bounded* iff for every nominal set X and $x \in FX$ there is an orbit-finite equivariant subset $Y \subseteq X$ with an equivariant inclusion function $i : Y \hookrightarrow X$ such that $x \in Fi[FY]$. If F preserves monics, the condition simplifies to $x \in FY \subseteq FX$.

lem:
finbounded

Lemma 5.1.6. *Monic preserving functors in Nom are finitary iff they are orbit-finitely bounded.*

Proof. Definition 5.1.5 is exactly the notion of “finitely bounded functor” as introduced in [2] instantiated to the category Nom , since Nom is a locally finitely presentable category (lfp) [9], where the finitely presentable objects are exactly the orbit-finite sets [8]. Thus, by the main result in [2], our orbit-finitely bounded functors are exactly the finitary functors on Nom if they preserve monics. $\square$

lem:
eqsepformulae

Lemma 5.1.7 (Equivariant Separating Formulae). *Let (C, γ) be an F -coalgebra, then there is an equivariant map $\psi : C \times C \rightarrow \mathcal{F}(\mathbb{A})$ such that*

$$d \models \psi(d, e) \quad \text{and} \quad e \not\models \psi(d, e)$$

for all $d, e \in C$ with $d \not\equiv_{\mathcal{F}(\mathbb{A})} e$.

Proof. Let us define ψ on orbits of $C \times C$, for every orbit O fix a representative d_O, e_O .

Thus, every $d, e \in C$ can be viewed as $(d, e) = (\pi \cdot d_O, \pi \cdot e_O)$ for some permutation π . Let $d \not\equiv_{\mathcal{F}(\mathbb{A})} e$, then by equivariance of logical indistinguishability also $d_O \not\equiv_{\mathcal{F}(\mathbb{A})} e_O$, which yields a formula φ_O , such that $d_O \models \varphi_O \wedge e_O \not\models \varphi_O$ or $d_O \not\models \varphi_O \wedge e_O \models \varphi_O$. Let us assume w.l.o.g. (using negation in the second case) that $d_O \models \varphi_O \wedge e_O \not\models \varphi_O$.

Now set

$$\psi(\pi \cdot d_O, \pi \cdot e_O) := \pi \cdot \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \}.$$

First, note that the conjunction is valid, since the underlying set is finitely supported by $\text{supp}(d_O) \cup \text{supp}(e_O)$ (if a permutation fixes d_O and e_O it can be merged with the σ in the definition) and is a subset of the orbit of φ_O . Next, we need to verify that the choice of π is irrelevant. Let π' be another permutation with $(d, e) = (\pi' \cdot d_O, \pi' \cdot e_O)$. First note that this means that $\pi^{-1} \cdot \pi' \cdot (d_O, e_O) = (d_O, e_O)$, thus the calculation goes as follows

$$\begin{aligned} \psi(\pi' \cdot d_O, \pi' \cdot e_O) &= \pi' \cdot \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \} \\ &= \bigwedge \{ \pi \cdot \pi^{-1} \cdot \pi' \cdot \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \} \\ &= \pi \cdot \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \}, \end{aligned}$$

where in the last step we reindex σ , since $\pi^{-1}\pi'$ fixes d_O and e_O .

Similarly, equivariance of ψ follows easily, for $\tau \in \mathfrak{S}_f(\mathbb{A})$:

$$\begin{aligned} \psi(\tau \pi \cdot d_O, \tau \pi \cdot e_O) &= \tau \pi \cdot \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \} \\ &= \tau \cdot \psi(\pi \cdot d_O, \pi \cdot e_O). \end{aligned}$$

Next, let us verify the required properties:

$$\begin{aligned} \pi \cdot d_O \models \psi(\pi \cdot d_O, \pi \cdot e_O) &\iff \pi \cdot d_O \models \pi \cdot \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \} \\ &\iff d_O \models \bigwedge \{ \sigma \cdot \varphi_O \mid \sigma \in \mathfrak{S}_f(\mathbb{A}) \wedge \sigma \cdot d_O = d_O \wedge \sigma \cdot e_O = e_O \}, \end{aligned}$$

thus we are left to check that d_O satisfies every $\sigma \cdot \varphi_O$ for $\sigma \cdot d_O = d_O$ and $\sigma \cdot e_O = e_O$, indeed by equivariance of the satisfaction relation, we have

$$d_O \models \sigma \cdot \varphi_O \iff \sigma^{-1} \cdot d_O \models \varphi_O \iff d_O \models \varphi_O,$$

which holds by the assumption on φ_O . A similar argument follows the property on e_O .

Finally, we are left to extend ψ to $d, e \in C$ where $d \equiv_{\mathcal{F}(\Lambda)} e$, in this case set $\psi(d, e) := \top$, which is equivariant by definition, thus finalizing the construction. $\square$

5.2 A Hennessy-Milner Theorem

Theorem 5.2.1 (Coalgebraic Hennessy-Milner Theorem). *If F is a finitary and monic pre-serving functor on Nom and Λ is separating, then the logic $\mathcal{F}(\Lambda)$ is expressive.*

Proof. Given two coalgebras $(X, \alpha), (Y, \beta)$ by Definition 2.3.10 and Lemma 4.2.9 it suffices to consider the sum $(C, \gamma) = (X, \alpha) + (Y, \beta)$ with $x, y \in C$. To show that $x \sim_F y$ follows from $x \equiv_{\mathcal{F}(\Lambda)} y$, we need to find a coalgebra morphism into a unifying coalgebra, we do this by quotienting C under $\equiv_{\mathcal{F}(\Lambda)}$, i.e., take the quotienting map

$$q : C \rightarrow C / \equiv_{\mathcal{F}(\Lambda)},$$

which sends every element of C to its equivalence class. We are left to show that this is indeed a coalgebra morphism into a coalgebra

$$(C / \equiv_{\mathcal{F}(\Lambda)}, \xi : C / \equiv_{\mathcal{F}(\Lambda)} \rightarrow F(C / \equiv_{\mathcal{F}(\Lambda)})).$$

Since the goal is for q be a coalgebra morphism, we define ξ as $\xi(q(c)) = Fq(\gamma(c))$, automatically satisfying the condition of coalgebra morphisms and equivariance follows from equivariance of q and γ .

We are left to show that ξ is well-defined, so let $c \equiv_{\mathcal{F}(\Lambda)} c'$, we have to show that $Fq(\gamma(c)) = Fq(\gamma(c'))$. By separation, it suffices to show that

$$Fq(\gamma(c)) \in \llbracket \heartsuit \rrbracket(Z) \iff Fq(\gamma(c')) \in \llbracket \heartsuit \rrbracket(Z),$$

for $\heartsuit \in \Lambda$ and $Z \in \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(C / \equiv_{\mathcal{F}(\Lambda)}))$.

By naturality of $\llbracket \heartsuit \rrbracket$ it suffices to show

$$\gamma(c) \in \llbracket \heartsuit \rrbracket(\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(q))(Z)) \iff \gamma(c') \in \llbracket \heartsuit \rrbracket(\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(q))(Z)). \quad (5.1) \quad \text{eq: gammatruthset}$$

Now, by Lemma 5.1.6 F is orbit-finitely bounded, thus we have orbit-finite equivariant subsets of C whose union we denote as $Y \subseteq C$ such that $\gamma(c), \gamma(c') \in FY \subseteq FC$.

Note that naturality of $\llbracket \heartsuit \rrbracket$ at the inclusion $i : Y \hookrightarrow C$ reads

$$\llbracket \heartsuit \rrbracket_C(u) \cap FY = \mathcal{Q}_{\text{fs}}(Fi)(\llbracket \heartsuit \rrbracket_C(u)) = \llbracket \heartsuit \rrbracket_Y(\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(i))(u)) \quad (5.2) \quad \text{eq: inclusion}$$

for every $u \in \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(C))$ (for instance $u := \text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(q))(Z)$).

Finally, assuming we have a function $g : \mathcal{Q}_{\text{fs}}(C / \equiv_{\mathcal{F}(\Lambda)}) \rightarrow \mathcal{F}(\Lambda)$, which maps each set of equivalence classes of states to a formula that defines it in Y , i.e.,

$$\llbracket g(W) \rrbracket_C \cap Y = q^{-1}[W] \cap Y \quad \text{for all } W \in \mathcal{Q}_{\text{fs}}(C / \equiv_{\mathcal{F}(\Lambda)})$$

or equivalently

$$\mathcal{Q}_{\text{fs}}(i) \circ \mathcal{Q}_{\text{fs}}(q) = \mathcal{Q}_{\text{fs}}(i) \circ \llbracket - \rrbracket_C \circ g. \quad (5.3) \text{ eq:separating}$$

Then, for any $d \in C$ with $\gamma(d) \in FY$, we have:

$$\begin{aligned} & \gamma(d) \in \llbracket \heartsuit \rrbracket_C (\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(q))(Z)) \\ \iff & \gamma(d) \in \llbracket \heartsuit \rrbracket_Y (\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(i))(\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(q))(Z))) & (5.2, \gamma(d) \in FY) \\ \iff & \gamma(d) \in \llbracket \heartsuit \rrbracket_Y (\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(i) \circ \mathcal{Q}_{\text{fs}}(q))(Z)) & (\text{functoriality of Ar}_{\heartsuit}) \\ \iff & \gamma(d) \in \llbracket \heartsuit \rrbracket_Y (\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(i) \circ \llbracket - \rrbracket_C \circ g)(Z)) & (5.3) \\ \iff & \gamma(d) \in \llbracket \heartsuit \rrbracket_Y (\text{Ar}_{\heartsuit}(\mathcal{Q}_{\text{fs}}(i))(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_C)(\text{Ar}_{\heartsuit}(g)(Z)))) & (\text{functoriality of Ar}_{\heartsuit}) \\ \iff & \gamma(d) \in \llbracket \heartsuit \rrbracket_C (\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_C)(\text{Ar}_{\heartsuit}(g)(Z))) & (5.2, \gamma(d) \in FY) \\ \iff & d \models \heartsuit(\text{Ar}_{\heartsuit}(g)(Z)). \end{aligned}$$

Thus (5.1) follows from $c \equiv_{\mathcal{F}(\Lambda)} c'$.

We are left to construct such a g . Recall from Lemma 5.1.7 that for $d \not\equiv_{\mathcal{F}(\Lambda)} e$ we have a $\psi(d, e)$ such that $d \models \psi(d, e)$ and $e \not\models \psi(d, e)$. Thus, we put

$$g(W) := \bigvee_{d \in Y, q(d) \in W} \bigwedge_{e \in Y, d \not\equiv_{\mathcal{F}(\Lambda)} e} \psi(d, e).$$

First, note that g is equivariant by construction, hence we are left to show that use of conjunction and disjunction are valid. Let us verify that the conjunction is correct, consider

$$\Psi_d := \{\psi(d, e) \mid e \in Y, d \not\equiv_{\mathcal{F}(\Lambda)} e\},$$

for every $d \in q^{-1}[W] \cap Y$. Note that equivariance of ψ and $\equiv_{\mathcal{F}(\Lambda)}$ gives $\pi \cdot \Psi_d = \Psi_{\pi \cdot d}$, which in turn yields that $\text{supp}(\Psi_d) \subseteq \text{supp}(d)$, thus Ψ_d is finitely supported. Furthermore, $\Psi_d \subseteq \psi[Y \times Y]$, thus it is an orbit-finite subset by Lemma 2.2.7, meaning $\Psi_d \in \mathcal{P}_{\text{of}}(\mathcal{F}(\Lambda))$, thus the conjunction $\bigwedge \Psi_d$ is indeed valid.

Next, we verify the disjunction, consider the assignment $\chi : Y \rightarrow \mathcal{F}(\Lambda)$ with $\chi(d) := \bigwedge \Psi_d$, which is an equivariant function by equivariance of Ψ_d . Then the disjunction can be written as $\bigvee \chi[q^{-1}[W] \cap Y]$, now, because χ is equivariant it suffices to show that $q^{-1}[W] \cap Y \in \mathcal{P}_{\text{of}}(Y)$, which of course holds, since $q^{-1}[W]$ is finitely supported by $\text{supp}(W)$.

Finally, its defining property follows from

$$y \models \chi(d) \iff y \equiv_{\mathcal{F}(\Lambda)} d \quad (y, d \in Y).$$

Indeed, if $y \equiv d$, then y satisfies every formula satisfied by d (and $d \models \chi(d)$ trivially). If $y \not\equiv d$, the conjunct $\psi(d, y)$ belongs to Ψ_d and is not satisfied by y . Therefore,

$$\begin{aligned} y \models g(W) & \iff \exists d \in q^{-1}[W] \cap Y, y \equiv d \\ & \iff q(y) \in W, \end{aligned}$$

which yields

$$\llbracket g(W) \rrbracket_C \cap Y = q^{-1}[W] \cap Y,$$

finishing the proof. □

Remark 5.2.2. The finitely supported powerset functor $\mathcal{P}_{\text{fs}}$, and thus also the neighborhood functor $\mathcal{N}_{\text{fs}}$ as well as the bag functor $\mathcal{B}_{\text{fs}}$ are not orbit-finitely bounded: For a finitely supported set X that is not orbit-finite, we have $X \in \mathcal{P}_{\text{fs}}(X)$, however there is no orbit-finite subset $Y \subseteq X$, such that $X \in \mathcal{P}_{\text{fs}}(Y)$, since this would imply $X = Y$.

We can however restrict all of these functors to their orbit-finite parts:

Definition 5.2.3 (Orbit-finite restriction functor). Consider any Nom functor F , we construct its orbit-finitely bounded restriction as

$$F_{\text{of}}(X) := \bigcup \{Fi[FY] \mid Y \subseteq X \text{ orbit-finite and equivariant subset with } i: Y \hookrightarrow X\}.$$

If F is additionally monic preserving, the construction simplifies to

$$F_{\text{of}}(X) := \bigcup \{FY \mid Y \subseteq X \text{ equivariant and orbit-finite subset}\}.$$

The orbit-finitely bounded restriction of a functor is intuitively a substructure of the original functor, let us make this notion concrete:

Definition 5.2.4 (Subfunctor). Let F be a functor on Nom, a subfunctor G of F is a functor that maps nominal sets X to equivariant subsets $G(X) \subseteq F(X)$, and for every $f: X \rightarrow Y$, $Ff[GX] \subseteq GY$.

Now, let us verify that the orbit-finitely bounded restriction is indeed a subfunctor.

Lemma 5.2.5. *For a monic preserving F the functor F_{of} is an orbit-finitely bounded subfunctor of F .*

Proof. F_{of} is orbit-finitely bounded by construction: take a nominal set X and $x \in F_{\text{of}}(X)$, then $x \in FY$ for some orbit-finite and equivariant subset $Y \subseteq X$.

Next, let us verify that F_{of} is a subfunctor of F , first we need to show that for all nominal sets X , $F_{\text{of}}(X) \subseteq F(X)$, which holds, since for every orbit-finite equivariant subset $Y \subseteq X$, $F(Y) \subseteq F(X)$.

Finally, let $f: X \rightarrow Y$, we need to verify that $Ff[F_{\text{of}}X] \subseteq F_{\text{of}}Y$. So let $y \in Ff[F_{\text{of}}X]$, i.e., there is some $x \in F_{\text{of}}X$, such that $Ff(x) = y$. We need to verify that $y = Ff(x) \in F_{\text{of}}Y$, i.e., there needs to be some orbit-finite equivariant subset $Y' \subseteq Y$ such that $Ff(x) \in FY'$. The assumption yields that there is an orbit-finite equivariant subset $X' \subseteq X$ such that $x \in FX'$. Of course, now we can just set $Y' := f[FX'] \subseteq Y$, then $y = Ff(x) \in Ff[FX'] = FY'$.

TODO: Make more readable.

□

6 Failure of the orbit-finite model property

6.1 Failure of the orbit-finite model property under global assumptions

In this chapter we aim to establish complexity bounds for reasoning in coalgebraic modal logics.

Crucially in Set, Schröder establishes a NEXPTIME upper bound on the satisfiability problem of coalgebraic modal logic under global assumptions using mild assumptions [11]. This is done by first establishing an exponential bound on the size of satisfying models under global assumptions.

Intuitively, in a nominal setting the exponential bound on states of a satisfying model translates to an exponential bound on the number of orbits of the model. Let us quickly show that such a property can not be established.

Definition 6.1.1 (Satisfiability under global assumptions). The *satisfiability problem under global assumptions* is to decide given two formulae φ, ψ whether there is a model C such that $C \models \varphi$, with a state $c \in C$, such that $C, c \models \psi$. We then say that ψ is satisfiable under the global assumption φ .

Lemma 6.1.2 (Failure of the orbit-finite model property under global assumptions). *There exists a formula ψ that is satisfiable under a global assumption φ such that no orbit-finite model witnesses the satisfiability.*

Proof. We give a counter example in binding Hennessy-Milner logic: Consider the formula $\psi := \bigwedge_{a \in \mathbb{A}} [a] \perp$, which is only satisfied in states that have no outgoing labeled transitions. We will now construct the global assumption φ in such a way that every state may introduce fresh names but must also remember previous names. The new name part is straightforward, the formula $\langle |a \rangle \langle a \rangle \top$ requires every state to have an outgoing binding transition. Next, consider the following formula:

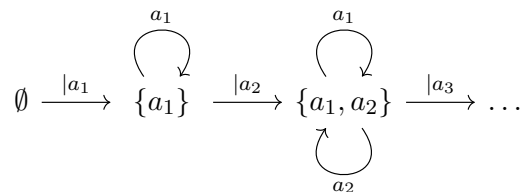
$$\bigwedge_{a, b \in \mathbb{A}, a \neq b} ((\langle b \rangle \top \rightarrow [|a] \langle b \rangle \top) \wedge ([b] \perp \rightarrow [|a] [b] \perp)),$$

which ensure that when introducing a new name via a $[a]$ -modality, that the set of previously remembered names stays intact.

The formula φ is thus given by

$$\varphi := \langle |a \rangle \langle a \rangle \top \wedge \bigwedge_{a, b \in \mathbb{A}, a \neq b} ((\langle b \rangle \top \rightarrow [|a] \langle b \rangle \top) \wedge ([b] \perp \rightarrow [|a] [b] \perp)).$$

Let us first consider a concrete satisfying model for these formulae, before proving that no orbit-finite model can exist. Consider the state space $S = \mathcal{P}_f(\mathbb{A})$ with transitions given by



which indeed satisfies $\emptyset \models \psi$ and $A \models \varphi$ for every $A \in \mathcal{P}_f(\mathbb{A})$. Since the support of the states grows with every step, the model is not orbit-finite.

Indeed, such a chain of states must be present in every satisfying model of ψ under the global assumption φ . For illustration, consider the equivariant function $E : C \rightarrow \mathcal{P}_{fs}(\mathbb{A})$ which maps each state to the set of outgoing labeled transitions, i.e.,

$$E(c) = \{a \in \mathbb{A} \mid c \models \langle a \rangle \top\}.$$

Now, let (C, γ) be such a satisfying model where $c_0 \in C$ satisfies ψ under the global assumption φ . By the definition of ψ this yields $E(c_0) = \emptyset$. Then, by global satisfaction of φ , there must be an $\langle a_1 \rangle$ -transition such that $\langle a_1 \rangle \top$ holds at the successor. We will call the successor c_1 . Furthermore, since $c_0 \models \varphi$, the set of outgoing labels of c_1 must be exactly $E(c_1) = \{a_1\}$.

Of course, repeating this construction yields an infinite chain of states $c_0, c_1, c_2, \dots$ such that $E(c_n) = \{a_1, \dots, a_n\} \subseteq \text{supp}(c_n)$. This works, because we can choose $a_{n+1} \# c_n$ in every step. Thus, there is an infinite chain of states with unbounded support sizes, which disproves the existence of an orbit-finite model. $\square$

6.2 Failure of the general orbit-finite model property

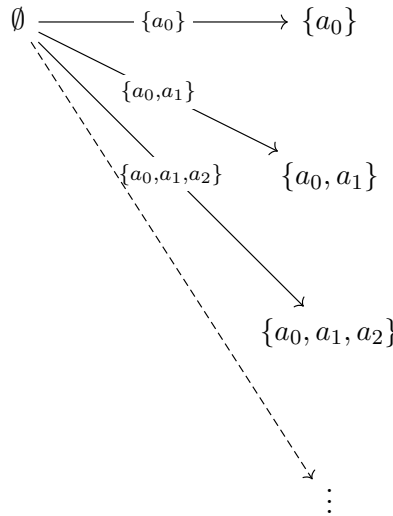
TODO: Rewrite to fit narrative, should be explained via the previous example.

In this section we will construct a nominal coalgebraic modal logic that admits a formula which does not admit an orbit-finite model. Thus, no orbit-finite model property holds generically for all nominal coalgebraic modal logics.

The idea builds on one fundamental fact about orbits. For two elements x, y in the same orbit, we have $x = \pi \cdot y$ for some permutation π . Thus, $\text{supp}(x) = \text{supp}(\pi \cdot y) = \pi \cdot \text{supp}(y)$. Hence, elements in the same orbit, share the same least support up to renaming. In our case the more interesting consequence is that $|\text{supp}(x)| = |\text{supp}(y)|$, i.e., elements in the same orbit have a same-sized support.

The route to construct a counterexample is thus clear: construct a formula that forces ever-growing support size.

Consider for example the following (orbit-) infinite model:



The states are finite sets of names and the transition labels likewise, thus let us fix the behaviour functor

$$FX = X^{\mathcal{P}_f(\mathbb{A})} \times \mathcal{P}_f(\mathbb{A}).$$

The idea is that an S -labeled transition forces the successor state to contain the names occurring in $S \in \mathcal{P}_f(\mathbb{A})$. A first intuitive idea would be to take a modality $[S]\varphi$, which holds if every S -successor satisfies φ . Using this modality together with a formula φ_S which says “the current state remembers all names in S ”, would only construct a counterexample if we could construct

$$\bigwedge_{S \in \mathcal{P}_f(\mathbb{A})} [S]\varphi_S.$$

However, $\mathcal{P}_f(\mathbb{A})$ is not orbit-finite. Thus, $[S]$ does not suffice.

Instead, let us take $[\exists a]\varphi$ with $\text{Ar}_{[\exists]}(X) = \mathbb{A} \times X$. $[\exists a]\varphi$ holds if every S -successor with $a \in S$ satisfies φ . Then, if we have a formula φ_a which holds if a is stored in the state, we can build

$$\bigwedge_{a \in \mathbb{A}} [\exists a]\varphi_a.$$

This indeed constitutes a formula that is only satisfied in non-orbit-finite models. We will verify this soon, let us first construct φ_a . This can easily be done by introducing a modality a with $\text{Ar}_a(X) = \mathbb{A}$ that intuitively holds if a is stored in the state.

Let us make this development more concrete now.

Example 6.2.1 (A logic suitable for forcing orbit-infinite models). Fix the behaviour functor

$$FX = X^{\mathcal{P}_f(\mathbb{A})} \times \mathcal{P}_f(\mathbb{A}),$$

together with modalities $[\exists a]$ and a with arities $\text{Ar}_{[\exists]}(X) = \mathbb{A} \times X$ and $\text{Ar}_a(X) = \mathbb{A}$.

F -coalgebras consist of a function $h : X \rightarrow (\mathcal{P}_f(\mathbb{A}) \rightarrow_{\text{fs}} X)$, where $h(c)(S)$ is the S -successor of c , and a function $N : X \rightarrow \mathcal{P}_f(\mathbb{A})$, where $N(c)$ is the finite set of atoms stored in c .

The expected semantics of our modalities are thus

$$\begin{aligned} c \models a &\iff a \in N(c) \\ c \models [\exists a]\varphi &\iff \forall S \in \mathcal{P}_f(\mathbb{A}). a \in S \Rightarrow h(c)(S) \models \varphi. \end{aligned}$$

These semantics are induced by the predicate liftings

$$\begin{aligned} \llbracket - \rrbracket_X &: \mathbb{A} \rightarrow \mathcal{P}_{\text{fs}}(FX) \\ \llbracket a \rrbracket_X &= \{ \langle k, M \rangle \mid a \in M \} \end{aligned}$$

and

$$\begin{aligned} \llbracket [\exists (-)] \rrbracket_X &: \mathbb{A} \times \mathcal{P}_{\text{fs}}(X) \rightarrow \mathcal{P}_{\text{fs}}(FX) \\ \llbracket [\exists a] \rrbracket_X(U) &= \{ \langle k, M \rangle \mid \forall S \in \mathcal{P}_f(\mathbb{A}). a \in S \Rightarrow k(S) \in U \}. \end{aligned}$$

Let us now verify equivariance and naturality of these predicate liftings.

For equivariance we have

$$\langle k, M \rangle \in \pi \cdot \llbracket a \rrbracket \iff a \in \pi^{-1} \cdot M \iff \pi(a) \in M \iff \langle k, M \rangle \in \llbracket \pi(a) \rrbracket,$$

and

$$\begin{aligned}
\langle k, M \rangle \in \pi \cdot \llbracket \exists a \rrbracket(U) &\iff \forall S. a \in S \Rightarrow (\pi^{-1} \cdot k)(S) \in U \\
&\iff \forall S. \pi(a) \in \pi \cdot S \Rightarrow k(\pi \cdot S) \in \pi \cdot U \\
&\iff \langle k, M \rangle \in \llbracket \exists \pi(a) \rrbracket(\pi \cdot U).
\end{aligned}$$

By equivariance the liftings are well-defined, since the results can be easily verified to land in $\mathcal{Q}_{\text{fs}}(FX)$.

Let $f : X \rightarrow Y$ be equivariant. The functor action is

$$Ff \langle k, M \rangle = \langle f \circ k, M \rangle.$$

Thus, for the nullary lifting,

$$\langle k, M \rangle \in \llbracket a \rrbracket_X \iff a \in M \iff Ff \langle k, M \rangle \in \llbracket a \rrbracket_Y.$$

Hence,

$$\llbracket a \rrbracket_X = (Ff)^{-1}[\llbracket a \rrbracket_Y].$$

For the box modality, consider $U \in \mathcal{P}_{\text{fs}}(Y)$,

$$\begin{aligned}
\langle k, M \rangle \in \llbracket \exists a \rrbracket_X(f^{-1}[U]) &\iff \forall S. a \in S \Rightarrow k(S) \in f^{-1}[U] \\
&\iff \forall S. a \in S \Rightarrow f(k(S)) \in U \\
&\iff Ff \langle k, M \rangle \in \llbracket \exists a \rrbracket_Y(U).
\end{aligned}$$

Therefore,

$$\llbracket \exists a \rrbracket_X(f^{-1}[U]) = (Ff)^{-1}[\llbracket \exists a \rrbracket_Y(U)].$$

Lemma 6.2.2. *Nominal coalgebraic modal logic does not admit a general orbit-finite model property.*

Proof. We proceed in the setting of the logic introduced in Example 6.2.1. Consider the formula

$$\varphi = \bigwedge_{a \in \mathbb{A}} [\exists a]a.$$

φ is satisfied by the (orbit-infinite) F -coalgebra $\mathcal{P}_{\text{f}}(\mathbb{A})$ with the trivial coalgebra structure $\gamma(S) = (id_{\mathcal{P}_{\text{f}}(\mathbb{A})}, S)$. Here every state stores exactly the names it consists of and the transition structure enforces that every S -successor is exactly S , thus trivially satisfying φ .

Let us now consider any satisfying coalgebra $(C, \langle h, N \rangle)$ with a $c \in C$ such that $c \models \varphi$. Since $c \models \varphi$, we have for every finite set $S \in \mathcal{P}_{\text{f}}(\mathbb{A})$ and every $a \in \mathbb{A}$:

$$\text{if } a \in S, \varphi \text{ yields } h(c)(S) \models a.$$

Hence, we have $S \subseteq N(h(c)(S))$, which implies that the set of stored names at a successor state contains the arbitrarily chosen input S . Finally, note that equivariance of N yields $\text{supp}(N(d)) \subseteq \text{supp}(d)$ for any $d \in C$, and since $N(d) \in \mathcal{P}_{\text{f}}(\mathbb{A})$ we have $\text{supp}(N(d)) = N(d)$. Thus, we obtain the crucial property

$$S \subseteq N(h(c)(S)) = \text{supp}(N(h(c)(S))) \subseteq \text{supp}(h(c)(S)).$$

That is, arbitrarily chosen $S \in \mathcal{P}_{\text{f}}(\mathbb{A})$ generate successor states whose supports contain S . Since the size of supports of states is unbounded, an orbit-finite model cannot exist. $\square$

6.3 Failure of the one-step orbit-finite model property

lem:one-ste
p-of-failure

Lemma 6.3.1 (Failure of the orbit-bounded one-step model property). *There exists a nominal coalgebraic modal logic that contains a satisfiable one-step formula that is not satisfiable by an orbit-finite model.*

Proof.

TODO: Incomplete in current state, needs adjustment from equivariant valuation to finitely supported one. Furthermore, one-step pair or one-step formula, which direction should this lemma go (if any)?

Put behavior

$$FX = X^{\mathcal{P}_f(\mathbb{A})},$$

with modality $[\exists a]$. Now for the counterexample take $V = \{v_a \mid a \in \mathbb{A}\}$ and the one-step pair

$$\varphi = \bigvee_{a \in \mathbb{A}} \neg v_a \quad \psi = \bigwedge_{a \in \mathbb{A}} [\exists a]v_a.$$

There is a satisfying model. Take

$$X = \mathcal{P}_f(\mathbb{A}) \quad \theta(v_a) = \{S \in \mathcal{P}_f(\mathbb{A}) \mid a \in S\},$$

and $t = \text{id}_X$. Every $S \in \mathcal{P}_f(\mathbb{A})$ satisfies φ , since $S \neq \mathbb{A}$. Furthermore, if $a \in S$, then

$$\text{id}_X(S) = S \in \theta(v_a).$$

Thus, $t \models \psi$, so $(\mathcal{P}_f(\mathbb{A}), \theta, \text{id}_X)$ is a satisfying model for (φ, ψ) , however it is not orbit-finite.

Let us show by contradiction that no orbit-finite model may exist. Suppose there is a model (X, θ, t) that satisfies the one-step pair (φ, ψ) such that X is orbit-finite. Consider the finitely supported name mapping

$$n : X \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(\mathbb{A}) \quad n(x) = \{a \mid x \in \theta(v_a)\}.$$

Indeed, since θ is finitely supported, we have $\text{supp}(n) = \text{supp}(\theta)$. Because X is orbit-finite there is an N such that $|\text{supp}(x)| \leq N$ for every $x \in X$, hence by equivariance we have

$$|\text{supp}(n(x))| \leq |\text{supp}(x)| + |\text{supp}(\theta)| \leq N + |\text{supp}(\theta)|.$$

Now, let T be a support of the finitely supported function $t : \mathcal{P}_f(\mathbb{A}) \rightarrow_{\text{fs}} X$ and choose an $S \in \mathcal{P}_f(\mathbb{A})$ such that

$$T \cup \text{supp}(\theta) \subseteq S \quad |S| > N + |\text{supp}(\theta)|.$$

Since $(X, \theta, t) \models \psi$, for every $a \in S$,

$$t(S) \in \theta(v_a),$$

and therefore

$$S \subseteq n(t(S)).$$

TODO:
Adjust for
 $\text{supp}(\theta)$.

Every finitely supported subset of $\mathbb{A}$ is either finite or cofinite, thus let us proceed by a case distinction

- If $n(t(S))$ is finite, then

$$S \subseteq n(t(S)) = \text{supp}(n(t(S))) \subseteq \text{supp}(t(S)),$$

which however contradicts $|S| > N$.

- Thus, $n(t(S))$ must be cofinite. By

$$\text{supp}(n(t(S))) = \mathbb{A} \setminus n(t(S))$$

we have

$$\mathbb{A} \setminus n(t(S)) = \text{supp}(n(t(S))) \subseteq \text{supp}(t(S)) \subseteq T \cup \text{supp}(S) = S.$$

But we previously established that $S \subseteq n(t(S))$, which only holds if $\mathbb{A} \setminus n(t(S)) = \emptyset$. This however cannot hold, since by φ , $n(t(S)) \neq \mathbb{A}$, a contradiction. $\square$

7 Sound and Complete Rulesets

TODO: Cite [7] and explain terminal chain method.

TODO: Briefly explain soundness and (weak/strong) completeness here.

7.1 Nominal Propositional Logic

Let us first study nominal propositional logic, which in our setting serves as the rank-0 logic.

It consists of the usual propositional connectives, with the difference being the inclusion of orbit-finite conjunctions instead of ordinary finite ones. For a fixed nominal set V of variables, we define the nominal set of propositional formulae $\text{Prop}(V)$ over V inductively by the conditions

- For every $v \in V$, $v \in \text{Prop}(V)$;
- when $\varphi \in \text{Prop}(V)$, so is $\neg\varphi$;
- for $\Phi \in \mathcal{P}_{\text{of}}(\text{Prop}(V))$ we have $\bigwedge \Phi \in \text{Prop}(V)$.

Definition 7.1.1 (Satisfaction of nominal propositional formulae). Let $\kappa : V \rightarrow_{\text{fs}} 2$ be a finitely supported truth assignment. Then, *satisfaction* is a finitely supported relation defined by

$$\begin{aligned} \kappa \models_{\text{PL}} v & : \iff \kappa(v) = \top \\ \kappa \models \neg\varphi & : \iff \kappa \not\models_{\text{PL}} \varphi \\ \kappa \models_{\text{PL}} \bigwedge \Phi & : \iff \forall \varphi \in \Phi. \kappa \models_{\text{PL}} \varphi. \end{aligned}$$

As in regular propositional logic, it is easy to see that every formula has a logically equivalent negation normal form (NNF) where negation only appears in front of variables. The same does not hold for conjunctive and disjunctive normal forms (CNF/DNF).

Lemma 7.1.2. *In nominal propositional logic, logically equivalent CNFs and equally DNFs need not exist.*

Proof. Fix the variable set $V = \{v_a \mid a \in \mathbb{A}\} \cup \{w_a \mid a \in \mathbb{A}\}$ and consider the formula

$$\varphi = \bigwedge_{a \in \mathbb{A}} (v_a \vee w_a),$$

which is in CNF. Let us assume that an equivalent DNF exists, i.e., a formula

$$\psi = \bigvee_{\Phi \in P} \bigwedge \Phi$$

with P and each $\Phi \in P$ orbit-finite, such that every Φ consists only of literals.

For every $S \in \mathcal{P}_f(\mathbb{A})$ we define valuations

$$\begin{aligned}\kappa_S(v_a) &= \top : \Longleftrightarrow a \in S \\ \kappa_S(w_a) &= \top : \Longleftrightarrow a \notin S.\end{aligned}$$

Indeed, fix $S \in \mathcal{P}_f(\mathbb{A})$, then $\kappa_S \models \varphi$ and thus $\kappa_S \models \psi$. Hence, $\kappa_S \models \bigwedge \Phi$ for some $\Phi \in P$. Since ψ is a DNF, each Φ consists of literal occurrences of v_a and w_a .

Let us now argue that $v_a \in \Phi$ if and only if $a \in S$. The forward direction follows since $\kappa_S \models \bigwedge \Phi$. Conversely, if there was an $a \in S$ such that v_a was not contained in Φ , we could define

$$\kappa'_S(v) = \begin{cases} \perp & \text{if } v = v_a \\ \kappa_S(v) & \text{otherwise.} \end{cases}$$

Then $\kappa'_S \models \bigwedge \Phi$, but $\kappa'_S \not\models \varphi$. Thus, every v_a with $a \in S$ is contained in Φ .

Using this fact we can establish that $S \subseteq \text{supp}(\Phi)$, which finishes the proof, since S can be chosen arbitrarily large to yield unbounded supports of the different $\Phi \in P$, which in turn shows that P is not orbit-finite.

Indeed, suppose $a \in S \setminus \text{supp}(\Phi)$ and pick $b \# S, \Phi$. Of course, $(a \ b) \cdot \Phi = \Phi$, thus $v_a \in \Phi$ implies $v_b \in \Phi$. This contradicts our previously established fact, since $v_b \in \Phi$ even though $b \notin S$.

TODO: Mention intuitive DNF

$$\bigvee_{S \in \mathcal{P}_{fs}(\mathbb{A})} \left(\bigwedge_{a \in S} v_a \wedge \bigwedge_{a \notin S} w_a \right).$$

□

Definition 7.1.3 (Propositional entailment). A formula $\psi \in \text{Prop}(V)$ is *propositionally entailed* by a finitely supported set of formulae Φ , written $\Phi \vdash_{\text{PL}} \psi$, if every valuation $\kappa : V \rightarrow_{fs} 2$ that satisfies all $\varphi \in \Phi$ also satisfies ψ .

In other settings a finitary notion of propositional entailment suffices. However, due to the use of orbit-finite conjunctions, not even orbit-finite compactness holds, which is why we opted for this (orbit-) infinitary version that is easily seen to be sound and strongly complete.

Lemma 7.1.4. *Nominal propositional logic is not orbit-finitely compact, i.e., there is a set $\Phi \in \mathcal{P}_{fs}(\text{Prop}(V))$ that is unsatisfiable, but all its orbit-finite subsets are satisfiable.*

TODO: Cite [4]

Proof. Let $V = \{v_a \mid a \in \mathbb{A}\}$ and put

$$\varphi_S = \left(\bigvee_{a \notin S} v_a \right) \wedge \left(\bigvee_{a \in S} \neg v_a \right),$$

for $S \in \mathcal{P}_f(\mathbb{A})$. Then take $\Phi = \{\varphi_S \mid S \in \mathcal{P}_f(\mathbb{A})\}$.

Let us verify that orbit-finite subsets of Φ are verifiable. First, note that elements of the same orbit have same-sized supports, thus for an orbit-finite subset $\Psi \subseteq \Phi$, there is an $n \in \mathbb{N}$ such that $|\text{supp}(\varphi_S)| \leq n$ for all $\varphi_S \in \Psi$. Now, choose a set $T \subseteq \mathbb{A}$ of $n+1$ atoms and define a valuation

$$\kappa_T(v_a) = \top : \Longleftrightarrow a \in T.$$

lem:
not-compact

Thus, for every $\varphi_S \in \Psi$ there is a true variable outside of S since $|T| = n + 1$ and there is a false variable outside of S , since κ_T only sets finitely many variables true. Hence, $\kappa_T \models \Psi$.

However, Φ is unsatisfiable. Let there be a finitely supported valuation $\kappa : V \rightarrow_{\text{fs}} 2$ such that $\kappa \models \Phi$. Then for $a, b \notin \text{supp}(\kappa)$ we have $(a \ b) \cdot \kappa = \kappa$, hence $\kappa(v_a) = \kappa(v_b)$, i.e., every variable outside $\text{supp}(\kappa)$ has the same truth value. However, $\varphi_{\text{supp}(\kappa)} \in \Phi$ a contradiction. $\square$

TODO: Motivate valuations.

Definition 7.1.5 (Propositional Models).

TODO: Adjust text, call (X, θ) propositional model.

Let $\theta : V \rightarrow \mathcal{P}_{\text{fs}} X$ be an equivariant valuation for a nominal set V of variables.

For a propositional formula $\varphi \in \text{Prop}(V)$ we denote by $\llbracket \varphi \rrbracket^0 \theta \in \mathcal{P}_{\text{fs}}(X)$ the equivariant function sending φ to its *truthset* under θ . We define $\llbracket \varphi \rrbracket^0 \theta$ by

$$\begin{aligned} \llbracket v \rrbracket^0 \theta &= \theta(v) \\ \llbracket \neg \varphi \rrbracket^0 \theta &= X \setminus \llbracket \varphi \rrbracket^0 \theta \\ \llbracket \bigwedge \Phi \rrbracket^0 \theta &= \bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket^0 \theta. \end{aligned}$$

TODO: Explain why valuations are equivariant and assignments finitely supported. Could use the example $V = X = \mathbb{A}$ with $\theta(a) = \{a\}$. Then valuation is equivariant, while evaluating a single point is fs.

Let us now show that these two notions of satisfiability are compatible. Given a model (X, θ) , we define

$$\kappa_x = \top \iff x \in \theta(v)$$

for $x \in X$. And for every nominal set of variables V we define the canonical model

$$\theta_V(v) = \{A \in \mathcal{P}_{\text{fs}}(V) \mid v \in A\}.$$

Lemma 7.1.6 (Compatibility of assignments and valuations). *Let V be a nominal set of variables and let $\varphi \in \text{Prop}(V)$.*

1. *For a given model (X, θ) and $x \in X$, κ_x is finitely supported and satisfies*

$$\kappa_x \models_{\text{PL}} \varphi \iff x \in \llbracket \varphi \rrbracket^0 \theta.$$

2. *For a given truth assignment $\kappa : V \rightarrow_{\text{fs}} 2$, we define $A_\kappa = \{v \in V \mid \kappa(v) = \top\}$. Then,*

$$\kappa \models_{\text{PL}} \varphi \iff A_\kappa \in \llbracket \varphi \rrbracket^0 \theta_V.$$

TODO: Prove.

TODO: Define substitutions.

Lemma 7.1.7. *If $\Psi \vdash_{\text{PL}} \varphi$ for $\varphi \in \text{Prop}(V)$, $\Psi \in \mathcal{P}_{\text{of}}(\text{Prop}(V))$, then*

$$\Psi \sigma \vdash_{\text{PL}} \varphi \sigma,$$

for every equivariant substitution $\sigma : V \rightarrow W$, where both V and W are nominal sets.

lem:prop-sub
st-entails

Proof.

TODO: Prove.

□

7.2 The One-Step Logic and Soundness

In coalgebraic modal logic, oftentimes properties of the full logic can be reduced to their *one-step logic*. We will adopt the same approach for nominal coalgebraic modal logic, thus let us define the corresponding one-step logic, i.e., the logic of rank-1 formulae.

Definition 7.2.1 (One-step formulae). Let Z be a nominal set, we put

$$\Lambda(Z) = \{\heartsuit(t) \mid \heartsuit \in \Lambda, t \in \text{Ar}_{\heartsuit}(Z)\}$$

and use the notation of Section 7.1 to define *one-step formulae* as elements of $\text{Prop}(\Lambda(V))$, for any nominal set V of variables.

Definition 7.2.2 (One-step semantics). Given an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}(X)$, we define the *extension* $\llbracket \varphi \rrbracket^1 \theta \subseteq FX$ of one-step formula φ by

$$\begin{aligned} \llbracket \neg \varphi \rrbracket^1 \theta &= FX \setminus \llbracket \varphi \rrbracket^1 \theta \\ \llbracket \bigwedge \Phi \rrbracket^1 \theta &= \bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket^1 \theta \\ \llbracket \heartsuit(t) \rrbracket^1 \theta &= \llbracket \heartsuit \rrbracket_X (\text{Ar}_{\heartsuit}(\theta)(t)) \end{aligned}$$

Definition 7.2.3 (One-step models). A *one-step model* is a triple (X, θ, t) consisting of a nominal set X , an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}} X$ and an element $t \in FX$.

We say that a one-step model (X, θ, t) *satisfies* a one-step formula φ , if

$$t \in \llbracket \varphi \rrbracket \theta$$

and write $(X, \theta, t) \models \varphi$. Furthermore, we say φ is *valid* over (X, θ) if $(X, \theta, t) \models \varphi$ for all $t \in FX$ and write $(X, \theta) \models \varphi$.

Definition 7.2.4 (One-step rules). A *one-step rule*

$$\frac{\varphi}{\psi}$$

consists of an arbitrary variable set V together with a rank-1 formula $\psi \in \text{Prop}(\Lambda(V))$, the *conclusion* and a propositional formula $\varphi \in \text{Prop}(V)$, the *premise*, which we additionally require being propositionally satisfiable.

A one-step rule φ/ψ with a variable set V is *one-step sound* if for every nominal set X and equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}} X$,

$$(X, \theta) \models_0 \varphi \implies (X, \theta) \models_1 \psi.$$

In other words $\llbracket \varphi \rrbracket^0 \theta = X \implies \llbracket \psi \rrbracket^1 \theta = FX$.

Let us now fix an equivariant set $\mathcal{R}$ of one-step sound one-step rules.

Definition 7.2.5 (Substitution). Let $\sigma : V \rightarrow W$ be an equivariant function on nominal sets V, W . We call σ a substitution and define its application to propositional formulae $\varphi \in \text{Prop}(V)$ by

$$\begin{aligned} v\sigma &= \sigma(v) & (v \in V) \\ (\neg\varphi)\sigma &= \neg(\varphi\sigma) & (\varphi \in \text{Prop}(V)) \\ (\bigwedge \Phi)\sigma &= \bigwedge \{\varphi\sigma \mid \varphi \in \Phi\} & (\Phi \in \mathcal{P}_{\text{of}}(\text{Prop}(V))) \end{aligned}$$

Similarly, substitution on one-step formulae $\psi \in \text{Prop}(\Lambda(V))$ is given by replacing the variable case by

$$(\heartsuit(v))\sigma = \heartsuit(\text{Ar}_{\heartsuit}(\sigma)(v)),$$

for $\heartsuit \in \Lambda$ and $v \in \text{Ar}_{\heartsuit}(V)$.

Definition 7.2.6. A (*substitution*) *instance* of a one-step rule φ/ψ over variables in V is a rule of the form

$$\frac{\varphi\sigma}{\psi\sigma},$$

where $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ is an equivariant substitution. The set $\text{Log}(\mathcal{R})$ of $\mathcal{R}$ -derivable formulae is the least set that is closed under propositional reasoning and under instances of rules in $\mathcal{R}$. Explicitly:

1. If $\Phi \subseteq \text{Log}(\mathcal{R})$ is finitely supported and $\Phi \vdash_{\text{PL}} \psi$, then $\psi \in \text{Log}(\mathcal{R})$; and
2. if $\varphi/\psi \in \mathcal{R}$ and σ is an equivariant substitution such that $\varphi\sigma \in \text{Log}(\mathcal{R})$, then $\psi\sigma \in \text{Log}(\mathcal{R})$.

We write $\vdash^{\mathcal{R}} \chi$ iff $\chi \in \text{Log}(\mathcal{R})$.

TODO: Maybe use definition lemma here, to use $\Phi \in \mathcal{P}_{\text{fs}}(\text{Log}(\mathcal{R}))$.

Lemma 7.2.7. $\text{Log}(\mathcal{R})$ is an equivariant subset of $\mathcal{F}(\Lambda)$ and thus a nominal set itself.

Proof. We need to show that $\psi \in \text{Log}(\mathcal{R}) \Rightarrow \pi \cdot \psi \in \text{Log}(\mathcal{R})$ for any permutation π and formula $\psi \in \mathcal{F}(\Lambda)$. Let us proceed by induction on the derivation of $\psi \in \text{Log}(\mathcal{R})$.

1. If $\psi \in \text{Log}(\mathcal{R})$ has been derived from a set of formulae Φ using propositional reasoning, then $\pi \cdot \psi$ can also be derived using propositional reasoning, by equivariance of propositional reasoning and the induction hypothesis which stipulates that $\pi \cdot \Phi \subseteq \text{Log}(\mathcal{R})$:

$$\Phi \vdash_{\text{PL}} \psi \Rightarrow \pi \cdot \Phi \vdash_{\text{PL}} \pi \cdot \psi.$$

2. If $\psi\sigma \in \text{Log}(\mathcal{R})$ has been derived from a rule $\varphi/\psi \in \mathcal{R}$ such that $\varphi\sigma \in \text{Log}(\mathcal{R})$, we first follow by equivariance of $\mathcal{R}$ that $\pi \cdot \varphi/\pi \cdot \psi \in \mathcal{R}$. Then, by the induction hypothesis and equivariance of substitution we can follow

$$\pi \cdot \varphi\sigma = (\pi \cdot \varphi)(\pi \cdot \sigma) \in \text{Log}(\mathcal{R}),$$

which in turn yields $(\pi \cdot \psi)(\pi \cdot \sigma) = \pi \cdot \psi\sigma \in \text{Log}(\mathcal{R})$. □

Definition 7.2.8. Let $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ be an equivariant substitution and (C, γ) an F -coalgebra. The *extension* of σ , $\llbracket \sigma \rrbracket_C : V \rightarrow \mathcal{P}_{\text{fs}} C$ is an equivariant function given by

$$\llbracket \sigma \rrbracket_C(v) = \llbracket \sigma(v) \rrbracket_C.$$

lem:subst

Lemma 7.2.9 (Substitution lemma). *For every equivariant substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ and an F -coalgebra (C, γ) , we have*

$$\begin{aligned} \llbracket \varphi \sigma \rrbracket_C &= \llbracket \varphi \rrbracket^0 \llbracket \sigma \rrbracket_C & (\varphi \in \text{Prop}(V)), \\ \llbracket \psi \sigma \rrbracket_C &= \gamma^{-1}[\llbracket \psi \rrbracket^1 \llbracket \sigma \rrbracket_C] & (\psi \in \text{Prop}(\Lambda(V))). \end{aligned}$$

Proof. We proceed by induction on φ and ψ respectively, where the negation and conjunction cases follow straightforwardly by the induction hypotheses. Thus, consider first the variable case. We have

$$\llbracket v \sigma \rrbracket_C = \llbracket \sigma(v) \rrbracket_C = \llbracket v \rrbracket^0 \llbracket \sigma \rrbracket_C.$$

Then, for a one-step formula $\heartsuit(v)$ with $v \in \text{Ar}_{\heartsuit}(V)$, we have

$$\begin{aligned} & \llbracket (\heartsuit(v)) \sigma \rrbracket_C \\ &= \llbracket \heartsuit(\text{Ar}_{\heartsuit}(\sigma)(v)) \rrbracket_C \\ &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_C(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_C)(\text{Ar}_{\heartsuit}(\sigma)(v)))] \\ &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_C(\text{Ar}_{\heartsuit}(\llbracket \sigma \rrbracket_C)(v))] \\ &= \gamma^{-1}[\llbracket \heartsuit(v) \rrbracket^1 \llbracket \sigma \rrbracket_C] \end{aligned}$$

□

thm:soundness

Theorem 7.2.10 (Soundness of nominal coalgebraic modal logic). *Every $\mathcal{R}$ -derivable formula is valid.*

Proof. We proceed by induction on the derivability relation $\vdash^{\mathcal{R}}$:

1. If ψ is a propositional consequence of a set $\Phi \subseteq \text{Log}(\mathcal{R})$, the induction hypothesis yields validity of every $\varphi \in \Phi$. Thus ψ must also be valid.
2. Let $\psi \sigma \in \text{Log}(\mathcal{R})$ have been derived from a rule $\varphi/\psi \in \mathcal{R}$ with variable set V , such that $\varphi \sigma \in \text{Log}(\mathcal{R})$. Let (C, γ) be an arbitrary F -coalgebra. The induction hypothesis yields validity of $\varphi \sigma$, by Lemma 7.2.9 this implies $(C, \llbracket \sigma \rrbracket) \models_0 \varphi$. Then, one-step soundness gives $(C, \llbracket \sigma \rrbracket) \models_1 \psi$, which again by Lemma 7.2.9 gives validity of $\psi \sigma$ under the arbitrary (C, γ) and thus validity in general. □

TODO: In Set, the set of all one-step sound one-step rules is one-step complete. This uses the canonical propositional model which probably will not work in our setting. Maybe develop a counterexample?

7.3 Rank-n Semantics and Completeness

TODO: Motivate final chain.

The rank of a modal formula is the nesting depth of its modal operators. Thus, we can describe the set of rank- n modal formulae by inductively defining the sets

$$\mathcal{F}(\Lambda)^0 = \text{Prop}(\emptyset) \quad \mathcal{F}(\Lambda)^{n+1} = \text{Prop}(\Lambda(\mathcal{F}(\Lambda)^n)).$$

We characterize this in the following lemma.

lem:Fn

Lemma 7.3.1. $\mathcal{F}(\Lambda)^n = \{\varphi \in \mathcal{F}(\Lambda) \mid \text{rk}(\varphi) \leq n\}$. In particular $\mathcal{F}(\Lambda) = \bigcup_{n \in \mathbb{N}} \mathcal{F}(\Lambda)^n$ and for every $\varphi \in \mathcal{F}(\Lambda)$ also $\varphi \in \mathcal{F}(\Lambda)^{\text{rk}(\varphi)}$.

Proof. Easy by induction on $\text{rk}(\varphi)$. □

TODO: Fix indexing, should numbers always be at top?

Definition 7.3.2 (Rank- n extension). Let us define the *rank- n extension* $\llbracket - \rrbracket_n : \mathcal{F}(\Lambda)^n \rightarrow \mathcal{P}_{\text{fs}}(F^n 1)$ inductively. At rank zero we interpret the boolean formulae over the empty variable set, thus we put

$$\begin{aligned}\llbracket \neg \varphi \rrbracket_0 &= 1 \setminus \llbracket \varphi \rrbracket_0 \\ \llbracket \bigwedge \Phi \rrbracket_0 &= \bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket_0.\end{aligned}$$

In the inductive case the extension $\llbracket - \rrbracket_{n+1} : \mathcal{F}(\Lambda)^{n+1} \rightarrow \mathcal{P}_{\text{fs}}(F^{n+1} 1)$ is defined by

$$\begin{aligned}\llbracket \neg \varphi \rrbracket_{n+1} &= F^{n+1} 1 \setminus \llbracket \varphi \rrbracket_{n+1} \\ \llbracket \bigwedge \Phi \rrbracket_{n+1} &= \bigcap_{\varphi \in \Phi} \llbracket \varphi \rrbracket_{n+1} \\ \llbracket \heartsuit(t) \rrbracket_{n+1} &= \llbracket \heartsuit \rrbracket_{F^n 1}(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_n)(t)).\end{aligned}$$

We write $\models^n \varphi$ if $\llbracket \varphi \rrbracket_n = F^n 1$.

Lemma 7.3.3 (Rank- n evaluation). *Let (C, γ) be an F -coalgebra and $\varphi \in \mathcal{F}(\Lambda)^n$. Then $\llbracket \varphi \rrbracket_C = \gamma_n^{-1}[\llbracket \varphi \rrbracket_n]$, where the rank- n component $\gamma_n : C \rightarrow F^n 1$ is defined by*

$$\begin{aligned}\gamma_0 &= !_C : C \rightarrow F^0 1 \\ \gamma_{n+1} &= F\gamma_n \circ \gamma : C \rightarrow F^{n+1} 1\end{aligned}$$

Proof. We proceed by induction on n , where the case $n = 0$ follows by a simple induction on propositional formulae. Now, let $\varphi \in \mathcal{F}(\Lambda)^{n+1}$, and let us proceed by induction on φ , where the propositional cases follow by the induction hypotheses. Thus consider $\varphi = \heartsuit(t)$ with $\heartsuit \in \Lambda, t \in \text{Ar}_{\heartsuit}(\mathcal{F}(\Lambda)^n)$. Then,

$$\begin{aligned}\llbracket \heartsuit(t) \rrbracket_C &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_C(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_C)(t))] \\ &= \gamma^{-1}[\llbracket \heartsuit \rrbracket_C(\text{Ar}_{\heartsuit}(\gamma_n^{-1} \circ \llbracket - \rrbracket_n)(t))] && \text{(Hypothesis)} \\ &= \gamma^{-1}[(F\gamma_n)^{-1}[\llbracket \heartsuit \rrbracket_{F^n 1}(\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_n)(t))]] && \text{(Naturality)} \\ &= (F\gamma_n \circ \gamma)^{-1}[\llbracket \heartsuit(t) \rrbracket_{n+1}] = \gamma_{n+1}^{-1}[\llbracket \heartsuit(t) \rrbracket_{n+1}]. && \square\end{aligned}$$

TODO: Needs motivation.

For the upcoming development we will assume that the behavior functor F admits a singleton F -coalgebra $e : 1 \rightarrow F1$.

TODO: For example $\mathbb{A} \times (-)$ does not admit e . But can this be fixed by taking $FX + 1$ and then pulling completeness back?

Example 7.3.4. All of our example behavior functors admit such a singleton F -coalgebra:

- $F = \mathcal{P}_{\text{fs}}$: put $e(*) = \{*\}$.
- $F = \mathcal{N}_{\text{fs}}$: put $e(*) = \{\{*\}\} \in \mathcal{P}_{\text{fs}}(\mathcal{P}_{\text{fs}}(1))$. This neighborhood family is also upward closed, so the same choice works for the monotone neighborhood subfunctor.

- $F = \mathcal{B}_{\text{fs}}$: put $e(*) = \mu$, where $\mu(*) = 0$.
- $F = \mathcal{D}_{\text{fs}}$: put $e(*) = \delta_*$, where $\delta_*(*) = 1$.
- $F = \mathcal{C}_{\text{fs}}$: put $e(*) = g$, where $g : \mathcal{Q}_{\text{fs}}(1) \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(1)$ is given by $g(U) = U$. This also satisfies $g(U) \subseteq U$, as required for the $\mathcal{C}_{ID}$ subfunctor.
- $FX = \mathcal{P}_{\text{fs}}(Z)$ for some fixed Z (as for propositional atoms): put $e(*) = \emptyset$.

All these examples are easily seen to be equivariant and well-defined.

Given a singleton F -coalgebra $e : 1 \rightarrow F1$ for any functor F , we construct the labelled singleton F -coalgebra $e^{\mathbb{A}} : 1 \rightarrow (F1)^{\mathbb{A}}$ by

$$e^{\mathbb{A}}(*) (a) = e(*) \quad (a \in \mathbb{A}).$$

Indeed, for every permutation π ,

$$(\pi \cdot e^{\mathbb{A}}(*))(a) = \pi \cdot e^{\mathbb{A}}(*) (\pi^{-1} \cdot a) = \pi \cdot e(*) = e(*).$$

Thus, $e^{\mathbb{A}}(*)$ is equivariant.

Furthermore, the binding singleton F -coalgebra $[\mathbb{A}]e : 1 \rightarrow [\mathbb{A}]F1$ is given by

$$[\mathbb{A}]e(*) = \langle a \rangle e(*),$$

where a is chosen arbitrarily i.e., $\langle a \rangle \{*\} = \langle b \rangle \{*\}$ trivially. Hence, $[\mathbb{A}]e(*)$ is well-defined.

Finally, pairing two singleton F -coalgebras $e : 1 \rightarrow F1$ and $d : 1 \rightarrow G1$ yields the product singleton $e : 1 \rightarrow F1 \times G1$, thus all of our running examples have this property.

Lemma 7.3.5. *For every n , the map*

$$e^n = F^n e : F^n 1 \rightarrow F^{n+1} 1$$

is an F -coalgebra structure such that its rank- n component as defined in Lemma 7.3.3 satisfies $e^n_n = \text{id}_{F^n 1}$.

Proof. Fix $n \in \mathbb{N}$. We show by induction on $k \leq n$ that

$$e^n_k = F^k (!_{F^{n-k} 1}) : F^n 1 \rightarrow F^k 1.$$

At $k = 0$, we have $e^n_0 = !_{F^n 1}$ by definition.

For $k < n$, the induction hypothesis and the observation-map recursion give

$$\begin{aligned} e^n_{k+1} &= F(e^n_k) \circ e^n \\ &= F^{k+1} (!_{F^{n-k} 1}) \circ F^n e \\ &= F^{k+1} (!_{F^{n-k} 1} \circ F^{n-k-1} e) \\ &= F^{k+1} (!_{F^{n-k-1} 1}), \end{aligned}$$

as claimed.

Finally, for $k = n$ we have $e^n_n = F^n (!_1) = \text{id}_{F^n 1}$. □

Corollary 7.3.6. *For every $\varphi \in \mathcal{F}(\Lambda)^n$,*

$$\llbracket \varphi \rrbracket_{(F^n 1, e^n)} = \llbracket \varphi \rrbracket_n \quad \text{and} \quad \models \varphi \iff \models^n \varphi.$$

Proof. By Lemma 7.3.3 and Lemma 7.3.5 we have

$$\llbracket \varphi \rrbracket_{(F^n 1, e^n)} = (e^n)^{-1} [\llbracket \varphi \rrbracket_n] = \llbracket \varphi \rrbracket_n.$$

Thus, $\models \varphi$ implies $\models_n \varphi$ using the F -coalgebra $(F^n 1, e^n)$. Conversely, if $\llbracket \varphi \rrbracket_n = F^n 1$, rank- n evaluation for any F -coalgebra (C, γ) gives $\llbracket \varphi \rrbracket_{(C, \gamma)} = \gamma_n^{-1} [\llbracket \varphi \rrbracket_n] = \gamma_n^{-1} [F^n 1] = C$. $\square$

Definition 7.3.7 (One-step derivability). A one-step formula $\psi \in \text{Prop}(\Lambda(V))$ is *one-step $\mathcal{R}$ -derivable over $(X, \theta : V \rightarrow \mathcal{P}_{\text{fs}} X)$* if ψ is propositionally entailed by one-step formulae of the form $\chi\sigma$, where $\sigma : W \rightarrow V$ is equivariant, $(\rho/\chi, W) \in \mathcal{R}$ and $\rho\sigma$ is valid over (X, θ) .

Definition 7.3.8 (One-step completeness). The ruleset $\mathcal{R}$ is *one-step complete* if for every (X, θ) , every one-step formula valid over (X, θ) is one-step $\mathcal{R}$ -derivable over (X, θ) .

Again, we will need a substitution lemma.

Lemma 7.3.9 (Substitution lemma for $\llbracket - \rrbracket_n$). *Let $n \in \mathbb{N}$ and let $\sigma : V \rightarrow \mathcal{F}(\Lambda)^n$ be an equivariant substitution. Put $\llbracket \sigma \rrbracket_n(v) = \llbracket \sigma(v) \rrbracket_n$, then*

$$\begin{aligned} \llbracket \varphi\sigma \rrbracket_n &= \llbracket \varphi \rrbracket_n^0 \llbracket \sigma \rrbracket_n & (\varphi \in \text{Prop}(V)), \\ \llbracket \psi\sigma \rrbracket_{n+1} &= \llbracket \psi \rrbracket_n^1 \llbracket \sigma \rrbracket_n & (\psi \in \text{Prop}(\Lambda(V))). \end{aligned}$$

Proof. As in the proof of Lemma 7.2.9, proceed by induction on φ and ψ , respectively. The variable case follows from the definition of $\llbracket \sigma \rrbracket_n$, and the propositional cases follow immediately from the induction hypotheses. For $\heartsuit \in \Lambda, t \in \text{Ar}_{\heartsuit}(V)$, we have

$$\begin{aligned} \llbracket (\heartsuit(t))\sigma \rrbracket_{n+1} &= \llbracket \heartsuit \rrbracket_{F^n 1} (\text{Ar}_{\heartsuit}(\llbracket - \rrbracket_n)(\text{Ar}_{\heartsuit}(\sigma)(t))) \\ &= \llbracket \heartsuit \rrbracket_{F^n 1} (\text{Ar}_{\heartsuit}(\llbracket \sigma \rrbracket_n)(t)) = \llbracket \heartsuit(t) \rrbracket_n^1 \llbracket \sigma \rrbracket_n. \end{aligned} \quad \square$$

Theorem 7.3.10 (Weak completeness of nominal coalgebraic modal logic). *Suppose $\mathcal{R}$ is one-step complete. Then for every formula φ ,*

$$\models \varphi \implies \vdash_{\mathcal{R}} \varphi.$$

Proof. By Lemma 7.3.1 and Corollary 7.3.6 it suffices to establish

$$\models_n \varphi \implies \vdash_{\mathcal{R}} \varphi,$$

for all $n \in \mathbb{N}$ and $\varphi \in \mathcal{F}(\Lambda)^n$. Hence, we proceed by induction on n , where the case $n = 0$ follows directly by the definition of propositional entailment.

Now, suppose $\models_{n+1} \varphi$ for $\varphi \in \mathcal{F}(\Lambda)^{n+1}$. Applying Lemma 7.3.9 with the identity substitution, yields one-step validity of φ over $(F^n 1, \llbracket - \rrbracket_n)$. Then, by one-step completeness, φ is propositionally entailed by one-step formulae of the form $\chi\sigma$ with $\sigma : W \rightarrow \mathcal{F}(\Lambda)^n$ equivariant and $\rho/\chi \in \mathcal{R}$ with variable set W such that $(F^n 1, \llbracket - \rrbracket_n) \models_0 \rho\sigma$. Again, by Lemma 7.3.9 we have $\models_n \rho\sigma$. Now, the induction hypothesis yields $\vdash_{\mathcal{R}} \rho\sigma$ and rule closure gives $\vdash_{\mathcal{R}} \chi\sigma$. Thus, by propositional reasoning we have $\vdash_{\mathcal{R}} \varphi$. $\square$

7.4 One-step complete rulesets

In this section we will introduce equivariant one-step sound and one-step complete rulesets for our running examples. Of course in each case we can follow soundness and weak-completeness for the full logic by Theorem 7.2.10 and Theorem 7.3.10.

In Set-based developments, one-step completeness is usually established for clausal representations of formulae, which exist since normally formulae have conjunctive/disjunctive normal forms. We can use a similar development, where we view the set of all clausal consequences as a representation of the formulas CNF.

Definition 7.4.1 (Clausal presentation). Let V be a nominal set of variables. The *clausal presentation* of a propositional formula $\varphi \in \text{Prop}(V)$ is an equivariant function

$$C : \text{Prop}(V) \rightarrow \mathcal{P}_{\text{fs}}(\text{Prop}(V))$$

defined by

$$C(\varphi) = \{ \bigwedge P \rightarrow \bigvee Q \mid P, Q \in \mathcal{P}_{\text{of}}(V), \varphi \vdash_{\text{PL}} \bigwedge P \rightarrow \bigvee Q \}.$$

Lemma 7.4.2 (Entailment by clausal presentations). *Every formula $\varphi \in \text{Prop}(V)$ is entailed by its clausal presentation, i.e.,*

$$C(\varphi) \vdash_{\text{PL}} \varphi.$$

Proof. We prove the contrapositive. Let $\kappa : V \rightarrow_{\text{fs}} 2$ be a valuation that falsifies φ and let $\text{Var}(\varphi) \in \mathcal{P}_{\text{of}}(V)$ be the equivariant closure of all variables appearing in φ . We need to show that κ falsifies a clause in $C(\varphi)$. Put $P = \{v \in \text{Var}(\varphi) \mid \kappa(v) = \top\}$ and $Q = \{v \in \text{Var}(\varphi) \mid \kappa(v) = \perp\}$. Since both P and Q are supported by $\text{supp}(\kappa)$, they are orbit-finite subsets of $\text{Var}(\varphi)$.

Now, consider the clause $\chi = \bigwedge P \rightarrow \bigvee Q$. By definition, $\kappa \models \bigwedge P$, but $\kappa \not\models \bigvee Q$, hence $\kappa \not\models \chi$. We are done once we establish $\chi \in C(\varphi)$, i.e., $\varphi \vdash_{\text{PL}} \chi$. Let us proceed by contraposition again. Suppose $\eta \not\models \chi$, then $\eta(v) = \kappa(v)$ for all $v \in \text{Var}(\varphi)$, hence $\eta \not\models \varphi$. $\square$

7.4.1 Hennessy-Milner logic

RK adjusted for orbit-finiteness:

$$(\text{RK}_{\text{of}}) \frac{\bigwedge_{w \in W} w \rightarrow v}{\bigwedge_{w \in W} [a]w \rightarrow [a]v}$$

for $W \in \mathcal{P}_{\text{of}}(V), v \in V, a \in \mathbb{A}$. Necessitation is the instance $W = \emptyset$.

Lemma 7.4.3. *The ruleset generated by RK_{of} is one-step sound.*

Proof. Fix the nominal variable set V , a subset $W \in \mathcal{P}_{\text{of}}(V)$ and $v \in V, a \in \mathbb{A}$. First, we show soundness of the ruleset, thus let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}(X)$. Then, validity of the premise is equivalent to

$$\bigcap_{w \in W} \theta(w) \subseteq \theta(v).$$

Similarly, validity of the conclusion reduces to

$$\bigcap_{w \in W} \llbracket [a]w \rrbracket^1 \theta \subseteq \llbracket [a]v \rrbracket^1 \theta$$

Now, by the definition of $\llbracket [a] \rrbracket$ it suffices to show that for every $f : \mathbb{A} \rightarrow_{\text{fs}} \mathcal{P}_{\text{fs}}(X)$, if $f(a) \subseteq \bigcap_{w \in W} \theta(w)$, then $f(a) \subseteq \theta(v)$. Which of course follows directly by the premise. $\square$

lem:
RKof-complete

Lemma 7.4.4. *The ruleset generated by RK_{of} is one-step complete.*

Proof. Note that by Lemma 7.4.2, every formula is entailed by their clausal presentation, thus it suffices to consider formulae

$$\chi = \bigwedge_{[a]w \in P} [a]w \rightarrow \bigvee_{[b]v \in Q} [b]v,$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$.

Then, let X be a nominal set and $\theta : V \rightarrow \mathcal{P}_{\text{fs}} X$ a valuation such that $(X, \theta) \models_1 \chi$. This validity corresponds to

$$\bigcap_{[a]w \in P} \llbracket [a]w \rrbracket^1 \theta \subseteq \bigcup_{[b]v \in Q} \llbracket [b]v \rrbracket^1 \theta.$$

Now, put $f(a) = \bigcap_{[a]w \in P} \theta(w)$. $f : \mathbb{A} \rightarrow \mathcal{P}_{\text{fs}} X$ is an element of the intersection, thus there must be a $[b]v \in Q$ such that

$$f(b) = \bigcap_{[b]w \in P} \theta(w) \subseteq \theta(v). \quad (*) \quad \text{itm: a-comp-lhs}$$

Hence, applying RK_{of} , where the premise is valid due to $(*)$, yields

$$\bigwedge_{[b]w \in P} [b]w \rightarrow [b]v,$$

which propositionally entails χ . $\square$

7.4.2 Binding Hennessy-Milner logic

Let us adapt the RK_{of} rule to the binding modality $|a$. For a formula $\llbracket [a]w_a \rrbracket \rightarrow \llbracket [a]v_a \rrbracket$ to hold, it should suffice to show that $w_b \rightarrow v_b$ holds for some $b \# a$. Then by the some/any property of freshness, $\llbracket [a]w_a \rrbracket \rightarrow \llbracket [a]v_a \rrbracket$ should be derivable.

The following derivation rule incorporates this:

$$(\text{RK}_{|a}) \frac{\bigwedge_{\substack{w \in W \\ b \# w}} (a \ b) \cdot w \rightarrow (a \ b) \cdot v}{\bigwedge_{w \in W} \llbracket [a]w \rrbracket \rightarrow \llbracket [a]v \rrbracket}$$

where V is a nominal set of variables, $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \in \mathbb{A}$ and $b \# a, W, v$.

The choice of this rule incorporates many design choices, first b needs to be fresh for W , otherwise the rule is unsound.

Lemma 7.4.5. *The ruleset generated by*

$$(\text{RK}_{|a}^{\sharp}) \frac{\bigwedge_{\substack{w \in W \\ b \# w}} (a \ b) \cdot w \rightarrow (a \ b) \cdot v}{\bigwedge_{w \in W} [a]w \rightarrow [a]v}$$

for $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \in \mathbb{A}$ and $b \# a, v$, where $b \# W$ need not hold, is unsound.

Proof.

TODO: Prove.

□

Furthermore, we allow W to contain elements whose support contains b , and then filter them in the conjunction in the premise. We could also postulate $b \# w$ for every $w \in W$ globally, however, together with $b \# W$, this would force W to be finite.

Lemma 7.4.6. *Let V be a nominal set and $W \in \mathcal{P}_{\text{of}}(V)$. If an atom b with $b \# W$ and $b \# w$ for all $w \in W$ exists, then W must be finite.*

Proof. It suffices to show that there is a uniform support on W , since orbit-finiteness of W then immediately yields finiteness. Thus, let us show that $\text{supp}(w) \subseteq \text{supp}(W)$ for every $w \in W$. Indeed, suppose there was $a \in \text{supp}(w) \setminus \text{supp}(W)$. Since $b \# W$ we have $(a \ b) \cdot W = W$, thus $(a \ b) \cdot w \in W$. But $a \in \text{supp}(w)$, hence $b \in (a \ b) \cdot (\text{supp}(w))$, which in turn yields

$$b \in \text{supp}((a \ b) \cdot w),$$

contradicting the assumption that $b \# w$ for every $w \in W$. Hence W is uniformly supported by $\text{supp}(W)$. □

Lemma 7.4.7. *The ruleset generated by $\text{RK}_{|a}$ is not one-step complete.*

Proof.

TODO: Maybe do not prove itself, the one-step completeness proof should motivate this.

□

Instead, consider the following rule

$$(\text{RK}_{|\text{of}}) \frac{\bigwedge_{\substack{p \in P \\ a \# p}} p @ a \rightarrow q @ a}{\bigwedge_{p \in P} [a]p \rightarrow [a]q}$$

for $P \in \mathcal{P}_{\text{of}}([\mathbb{A}]V)$, $q \in [\mathbb{A}]V$ and $a \# P, q$.

Lemma 7.4.8. *The $\text{RK}_{|\text{of}}$ rule entails $\text{RK}_{|a}$.*

Proof. Let V be a nominal set of variables, $W \in \mathcal{P}_{\text{of}}(V)$, $a \in \mathbb{A}$, $v \in V$ and $b \# a, W, v$. Furthermore, let

$$\bigwedge_{\substack{w \in W \\ b \# w}} (a \ b) \cdot w \rightarrow (a \ b) \cdot v$$

hold. Now, we instantiate $\text{RK}_{|\text{of}}$ with $P = \{\langle a \rangle w \mid w \in W\}$ and $q = \langle a \rangle v$. Then, $b \# P, q$, thus we may derive

$$\bigwedge_{\langle a \rangle w \in P} [[a]w \rightarrow [a]v],$$

which is the required conclusion, since $\langle a \rangle w \in P \iff w \in W$. $\square$

lem:
RKbind-sound

Lemma 7.4.9. *The ruleset generated by $\text{RK}_{|\text{of}}$ is one-step sound for the binding modality $[-]$.*

Proof. For one-step soundness consider a nominal set V of variables with $P \in \mathcal{P}_{\text{of}}([\mathbb{A}]V)$, $q \in [\mathbb{A}]V$ and some $a \# P, q$. Furthermore, let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$. Validity of the premise of $\text{RK}_{|\text{of}}$ reduces to

$$\bigcap_{\substack{p \in P \\ a \# p}} \theta(p @ a) \subseteq \theta(q @ a). \quad (*) \quad \text{itm:} \\ \text{premise-inclu} \\ \text{sion-binding}$$

Similarly, validity of the conclusion reduces to

$$\bigcap_{p \in P} [[[[-]p]]^1 \theta \subseteq [[[[-]q]]^1 \theta.$$

Thus, by the definition of $[[[-]]]$, it suffices to show that for every $R \in [\mathbb{A}]\mathcal{P}_{\text{fs}}(X)$ and $b \# a, P, R, q$, if for all $p \in P$ with $b \# p$, $R @ b \subseteq \theta(p @ b)$, then $R @ b \subseteq \theta(q @ b)$. Since $a, b \# P, q$, applying $(a \ b)$ to $(*)$ yields

$$\bigcap_{\substack{p \in P \\ b \# p}} \theta(p @ b) \subseteq \theta(q @ b),$$

hence finishing the proof of one-step soundness. $\square$

lem:RKbind-c
complete

Lemma 7.4.10. *The ruleset generated by $\text{RK}_{|\text{of}}$ is one-step complete for the binding modality $[-]$.*

Proof. Let us now turn to the proof of one-step completeness. Let us again consider only the clausal presentations, i.e., formulae of the form

$$\chi = \bigwedge_{[-]p \in P} [[[-]p] \rightarrow \bigvee_{[-]q \in Q} [[[-]q],$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$. Then, let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$, such that $(X, \theta) \models_1 \chi$. By this validity we have

$$\bigcap_{[-]p \in P} [[[[-]p]]^1 \theta \subseteq \bigcup_{[-]q \in Q} [[[[-]q]]^1 \theta.$$

Now, choose an $a \# P$ and put $R = \langle a \rangle \bigcap_{\substack{[-]p \in P \\ a \# p}} \theta(p @ a)$. It is easy to see that the intersection is supported by $\text{supp}(P) \cup \{a\}$, hence $\text{supp}(R) \subseteq \text{supp}(P)$. Thus, we have

$$R @ b = \bigcap_{\substack{[-]p \in P \\ b \# p}} \theta(p @ b),$$

for every $b \# P$. By the semantics of $[-]$, this yields $R \in \bigcap_{[-]p \in P} \llbracket [-]p \rrbracket^1 \theta$ and consequently also $R \in \bigcup_{[-]q \in Q} \llbracket [-]q \rrbracket^1 \theta$. Thus, there must be an $[-]q \in Q$ such that for any $b \# P, q$ we have

$$R @ b = \bigcap_{\substack{[-]p \in P \\ b \# p}} \theta(p @ b) \subseteq \theta(q @ b).$$

Using this we can derive

$$\bigwedge_{[-]p \in P} [-]p \rightarrow [-]q,$$

from RK_{of} , which propositionally entails χ . $\square$

Corollary 7.4.11. *The ruleset for binding HML generated by the rules RK_{of} and RK_{of} is one-step sound and one-step complete.*

TODO: Prove fusion

7.4.3 Deallocation Logic

Let us first develop a rule for the deallocation modality. Consider the formula $[a]w \rightarrow [a]v$. This formula should hold whenever, $a \# w, v$ and $w \rightarrow v$ holds. Thus, an orbit-finite adaptation might be

$$(\text{RK}_{a|}) \frac{\bigwedge_{w \in W} w \rightarrow v}{\bigwedge_{w \in W} [a]w \rightarrow [a]v}$$

for a nominal set V and $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$ and $a \in \mathbb{A}$ with $a \# v$ and $a \# w$ for every $w \in W$.

However, this naive rule does not yield a one-step complete ruleset. Indeed, a proof of one-step completeness would proceed as in Lemma 7.4.4, i.e, we would first assume

$$\bigcap_{[a]w \in P} \llbracket [a]w \rrbracket^1 \theta \subseteq \bigcup_{[b]v \in Q} \llbracket [b]v \rrbracket^1 \theta,$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$ to hold. Then, one would want to define a naive function $f \in R(\mathcal{P}_{\text{fs}}(X))$ such that $f \in \llbracket [a]w \rrbracket^1 \theta$ for every $[a]w \in P$. However, the naive choice of $f(a) = \bigcap_{[a]w \in P} \theta(w)$ does not suffice, since $a \# f(a)$ need not hold and thus $f \in R(\mathcal{P}_{\text{fs}}(X))$ can not be proven.

This motivates the following counterexample to one-step completeness.

Example 7.4.12 ($\text{RK}_{a|}$ is not one-step complete). Take $V = \{w_a \mid a \in \mathbb{A}\} \cup \{v\}$. Then,

$$\chi = \bigwedge_{b \neq a} [a]w_b \rightarrow [a]v$$

is valid under $X = \mathbb{A}$ and $\theta(w_b) = \mathbb{A} \setminus \{b\}$, $\theta(v) = \emptyset$. Because for an $f \in R(\mathcal{P}_{\text{fs}}\mathbb{A})$ that satisfies the antecedent, we have

$$f(a) \subseteq \bigcap_{b \neq a} \theta(w_b) = \{a\}$$

and since $a \# f(a)$, we have $f(a) = \emptyset$, thus satisfying $[a]v$.

However, take the equivariant substitution $\sigma([c]u) = u$ for all $[c]u \in \Lambda(V)$. Then, if χ is propositionally entailed by a finitely supported set of formulae, so is $\chi\sigma$ due to Lemma 7.1.7. However,

$$\chi\sigma = \bigwedge_{b \neq a} w_b \rightarrow v_a$$

is not valid, since it is not satisfied by a . Thus, χ is not one-step derivable over $(\mathbb{A}, \theta)$.

The issue is that $a \# f(a)$ needs to hold for our semantics, which the current rule does not express. Hence, we can amend this by removing a from the support in the premise. Consider the rule

$$\text{(RK}_{\text{of}}\text{)} \frac{(\bigwedge_{w \in W} w \wedge \bigwedge_{\substack{w \in W \\ b \# W}} (a \ b) \cdot w) \rightarrow v}{\bigwedge_{w \in W} [a]w \rightarrow [a]v}$$

TODO: Use Pitts' restriction here!

for a nominal set V with $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \# v$ and $a \# w$ for every $w \in W$.

Lemma 7.4.13. *The rule RK_{of} entails RK_a .*

Proof. This is easy to see, since $\bigwedge_{w \in W} w \rightarrow v$ propositionally entails

$$(\bigwedge_{w \in W} w \wedge \bigwedge_{\substack{w \in W \\ b \# W}} (a \ b) \cdot w) \rightarrow v,$$

for every nominal set V and $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \# v$ with $a \# w$ for every $w \in W$. $\square$

Lemma 7.4.14. *The ruleset generated by RK_{of} is one-step sound for deallocating modalities $[a]$.*

Proof. We proceed by a similar argument as in Lemma 7.4.3. Fix a nominal variable set V , a subset $W \in \mathcal{P}_{\text{of}}(V)$ and $v \in V$, $a \in \mathbb{A}$ such that $a \# v$ and $a \# w$ for every $w \in W$. Then, let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$. Validity of the premise means that

$$(\bigcap_{w \in W} \theta(w) \cap \bigcap_{\substack{w \in W \\ b \# W}} \theta((a \ b) \cdot w)) \subseteq \theta(v).$$

We need to show that

$$\bigcap_{w \in W} \llbracket [a]w \rrbracket^1 \theta \subseteq \llbracket [a]v \rrbracket^1 \theta.$$

Indeed, let $f \in R(\mathcal{P}_{\text{fs}}X)$, with $f(a) \subseteq \bigcap_{w \in W} \theta(w)$. By the premise we are done once we establish

$$f(a) \subseteq \bigcap_{\substack{w \in W \\ b \# W}} \theta((a \ b) \cdot w).$$

Thus, fix $w \in W$ and $b \# W$. Now, let us choose a $c \# a, b, W, w, f(a)$, then $(b \ c) \cdot w \in W$, since $c, b \# W$. Hence, equivariance of θ and $a, c \# f(a)$ yield

$$f(a) = (a \ c) \cdot f(a) \subseteq (a \ c) \cdot \theta((b \ c) \cdot w) = \theta((a \ c)(b \ c) \cdot w) = \theta((a \ b) \cdot w),$$

where the last step uses $a, c \# w$. $\square$

Lemma 7.4.15. *The ruleset generated by RK_{of} is one-step complete for the modalities $[a]$.*

Proof. By Lemma 7.4.2 it suffices to consider formulae of the form

$$\chi = \bigwedge_{[a]w \in P} [a]w \rightarrow \bigvee_{[b]v \in Q} [b]v,$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$. Let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$ such that $(X, \theta) \models_1 \chi$. By this validity we have

$$\bigcap_{[a]w \in P} \llbracket [a]w \rrbracket^1 \theta \subseteq \bigcup_{[b]v \in Q} \llbracket [b]v \rrbracket^1 \theta.$$

Now, put $W_a = \{w \mid [a]w \in P\}$ for every $a \in \mathbb{A}$ and note that $\text{supp}(W_a) \subseteq \text{supp}(P) \cup \{a\}$ by simple calculations. Then, take

$$f(a) = \llbracket \bigwedge_{w \in W_a} w \wedge \bigwedge_{\substack{w \in W_a \\ b \# W_a}} (a \ b) \cdot w \rrbracket^0 \theta.$$

The extra conjuncts yield $a \# f(a)$ as required. To see this, let us verify that $f(a)$ is supported by $\text{supp}(W_a) \setminus \{a\}$ for any $a \in \mathbb{A}$. Note first, that we can equivalently state $f(a)$ as

$$f(a) = \bigcap_{\substack{w \in W_a \\ b \notin \text{supp}(W_a) \setminus \{a\}}} \theta((a \ b) \cdot w).$$

Now, let π be a permutation that fixes $\text{supp}(W_a) \setminus \{a\}$, then

$$\begin{aligned} \pi \cdot f(a) &= \bigcap_{\substack{w \in W_a \\ b \notin \text{supp}(W_a) \setminus \{a\}}} \theta(\pi \cdot (a \ b) \cdot w) \\ &= \bigcap_{\substack{w \in W_a \\ b \notin \text{supp}(W_a) \setminus \{a\}}} \theta((a \ \pi(b)) \cdot w) \\ &= \bigcap_{\substack{w \in W_a \\ \pi(b) \notin \text{supp}(W_a) \setminus \{a\}}} \theta((a \ \pi(b)) \cdot w) = f(a), \end{aligned}$$

where the second equality follows, because $\pi \cdot (a \ b) \cdot W_a = (a \ \pi(b)) \cdot W_a$ for any $b \notin \text{supp}(W_a) \setminus \{a\}$. Hence, $a \# f(a)$. Furthermore, f itself is supported by $\text{supp}(P)$. Let σ be a permutation fixing $\text{supp}(P)$, then $\sigma \cdot W_a = W_{\sigma(a)}$ by definition of W_a . Now,

$$\begin{aligned} \sigma \cdot f(a) &= \bigcap_{\substack{w \in W_a \\ b \notin \text{supp}(W_a) \setminus \{a\}}} \theta((\sigma(a) \ \sigma(b)) \cdot (\sigma \cdot w)) \\ &= \bigcap_{\substack{w \in W_{\sigma(a)} \\ b \notin \text{supp}(W_{\sigma(a)}) \setminus \{\sigma(a)\}}} \theta((\sigma(a) \ b) \cdot w) = f(\sigma(a)), \end{aligned}$$

where the first equality follows since generally $\sigma \circ (a \ b) \circ \sigma^{-1} = (\sigma(a) \ \sigma(b))$. Thus,

$$(\sigma \cdot f)(a) = \sigma \cdot f(\sigma^{-1}(a)) = f(a),$$

which yields $f \in R(\mathcal{P}_{\text{fs}}X)$ as required.

Then, by definition $f \in \llbracket [a]w \rrbracket^1 \theta$ for every $a \in \mathbb{A}$ and $w \in W_a$, thus there must be a $[a]v \in Q$ such that $f \in \llbracket [a]v \rrbracket^1 \theta$. Finally, this yields $f(a) \subseteq \theta(v)$, which yields validity of the premise of RK_{of} , thus deriving

$$\bigwedge_{w \in W_a} [a]w \rightarrow [a]v,$$

which in turn propositionally entails χ . $\square$

Now, for the unknown transition assume we want to derive a formula $\llbracket [a]w \rrbracket \rightarrow \llbracket [a]v \rrbracket$ where $a \# w, v$. In that case it simply suffices to derive $w \rightarrow v$. Thus, consider the rule

$$(\text{RK}_{|a|}) \frac{\bigwedge_{w \in W} w \rightarrow v}{\bigwedge_{w \in W} \llbracket [a]w \rrbracket \rightarrow \llbracket [a]v \rrbracket}$$

for a nominal set V with $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$ and $a \# v$ as well as $a \# w$ for every $w \in W$.

Again, this rule does not by itself suffice to prove one-step completeness. Hence consider the following generalization

$$(\text{RK}_{|\text{of}|}) \frac{\bigwedge_{w \in W} w \rightarrow v}{\bigwedge_{\substack{w \in W \\ b \in \mathbb{A} \\ b \# w}} \llbracket [b]w \rrbracket \rightarrow \llbracket [a]v \rrbracket}$$

for a nominal set V and $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$ and $a \# v$.

Lemma 7.4.16. *The rule $\text{RK}_{|\text{of}|}$ entails $\text{RK}_{|a|}$.*

Proof. Fix a nominal set V with $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \# v$ and $a \# w$ for every $w \in W$. Trivially,

$$\bigwedge_{\substack{w \in W \\ b \in \mathbb{A} \\ b \# w}} \llbracket [b]w \rrbracket \rightarrow \llbracket [a]v \rrbracket,$$

reduces to

$$\bigwedge_{w \in W} \llbracket [a]w \rrbracket \rightarrow \llbracket [a]v \rrbracket,$$

since $a \# w$ for every $w \in W$. $\square$

Lemma 7.4.17. *The ruleset generated by $\text{RK}_{|\text{of}|}$ is one-step sound for the modalities $\llbracket [a] \rrbracket$.*

Proof. Fix a nominal set V of variables with $W \in \mathcal{P}_{\text{of}}(V)$, $v \in V$, $a \# v$. Then, let X be a nominal set with equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}} X$, such that $(X, \theta) \models^0 \bigwedge_{w \in W} w \rightarrow v$. Hence,

$$\bigcap_{w \in W} \theta(w) \subseteq \theta(v).$$

We need to prove validity of the conclusion, i.e.,

$$\bigcap_{\substack{w \in W \\ b \in \mathbb{A} \\ b \# w}} \llbracket [b]w \rrbracket^1 \theta \subseteq \llbracket [a]v \rrbracket^1 \theta.$$

Thus, let $H \in \mathcal{P}_{\text{fs}} X$ such that $H \subseteq \theta(w)$ for every $w \in W$, then $H \subseteq \theta(v)$ by the premise. $\square$

Lemma 7.4.18. *The ruleset generated by $\text{RK}_{|\text{of}|}$ is one-step complete for unknown modalities $[[a]]$.*

Proof. By Lemma 7.4.2 it suffices to consider formulae of the form

$$\chi = \bigwedge_{[[a]]w \in P} [[a]]w \rightarrow \bigvee_{[[b]]v \in Q} [[b]]v,$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$. Let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$ such that $(X, \theta) \models_1 \chi$. By this validity we have

$$\bigcap_{[[a]]w \in P} [[[[a]]w]]^1 \theta \subseteq \bigcup_{[[b]]v \in Q} [[[[b]]v]]^1 \theta.$$

Now, put $H = \bigcap_{[[a]]w \in P} \theta(w)$, then there must be a $[[b]]v \in Q$ such that $H \subseteq \theta(v)$. This yields validity of

$$\bigwedge_{w \in W} w \rightarrow v.$$

for $W = \{w \mid \exists a \in \mathbb{A}. [[a]]w \in P\}$.

Now, since $b \# v$, we can derive

$$\bigwedge_{\substack{w \in W \\ c \in \mathbb{A} \\ c \# w}} [[c]]w \rightarrow [[b]]v,$$

using $\text{RK}_{|\text{of}|}$. Finally, since $[[a]]w = [[c]]w$ for any $a, c \# w$ and since for every $w \in W$ such a fresh atom exists by definition, we can equivalently state the formula as

$$\bigwedge_{[[a]]w \in P} [[a]]w \rightarrow [[b]]v,$$

which propositionally entails χ . □

7.4.4 Neighborhood logic

To construct a generic congruence rule over all possible modalities, let us first fix an orbit-finite equivariant equivalence relation $E \subseteq V \times V$, which induces a quotient $q_E : V \rightarrow V/\sim_E$. The congruence rule can then be defined as

$$(\text{cong}_{\heartsuit}) \frac{\bigwedge_{(v,w) \in E} (v \leftrightarrow w)}{\heartsuit(s) \leftrightarrow \heartsuit(t)},$$

where $s, t \in \text{Ar}_{\heartsuit}(V)$ and $\text{Ar}_{\heartsuit}(q_E)(t) = \text{Ar}_{\heartsuit}(q_E)(s)$.

Lemma 7.4.19 (Soundness of $\text{cong}_{\heartsuit}$). *For every set of modal operators Λ , the ruleset generated by $\text{cong}_{\heartsuit}$ for $\heartsuit \in \Lambda$ is one-step sound.*

Proof. Let V be a nominal set of variables and $E \subseteq V \times V$ an equivariant equivalence relation. Then, let X be a nominal set with an equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$. Validity of the premise under (X, θ) means

$$\forall (v, w) \in E. \theta(v) = \theta(w),$$

lem:
cong-sound

hence θ factors through the quotient q_E .

We can use this directly to prove the conclusion, let $s, t \in \text{Ar}_\heartsuit(V)$ with

$$\text{Ar}_\heartsuit(q_E)(t) = \text{Ar}_\heartsuit(q_E)(s). \quad (*) \text{ itm:cong-eq}$$

Then,

$$\llbracket \heartsuit(s) \rrbracket^1 \theta = \llbracket \heartsuit \rrbracket(\text{Ar}_\heartsuit(\theta)(s)) \stackrel{(*)}{=} \llbracket \heartsuit \rrbracket(\text{Ar}_\heartsuit(\theta)(t)) = \llbracket \heartsuit(t) \rrbracket^1 \theta. \quad \square$$

Lemma 7.4.20 (Completeness of neighborhood logic). *Let*

lem:
cong-complete

$$(\text{cong}_a) \frac{v \leftrightarrow w}{[a]v \leftrightarrow [a]w},$$

for $a \in \mathbb{A}$ as well as a nominal set V of variables with $v, w \in V$ generate a ruleset for neighborhood logic. It is easy to see that this is equivalent to the one generated by $\text{cong}_\heartsuit$.

The ruleset generated by cong_a is one-step complete for neighborhood logic.

Proof. Let V be a nominal set of variables. As for HML, it suffices to consider a formula

$$\chi = \bigwedge_{[a]w \in P} [a]w \rightarrow \bigvee_{[b]v \in Q} [b]v,$$

for $P, Q \in \mathcal{P}_{\text{of}}(\Lambda(V))$. Let $(X, \theta) \models_1 \chi$ for some nominal set X and equivariant valuation $\theta : V \rightarrow \mathcal{P}_{\text{fs}}X$. Recall that this yields

$$\bigcap_{[a]w \in P} \llbracket [a]w \rrbracket^1 \theta \subseteq \bigcup_{[b]v \in Q} \llbracket [b]v \rrbracket^1 \theta.$$

This time we need to define a function $f : \mathbb{A} \rightarrow \mathcal{N}_{\text{fs}}X$. Put $f(a) = \{\theta(w) \mid [a]w \in P\}$. Again, f is an element of the intersection, thus there must be a $[b]v \in Q$ such that

$$\theta(v) \in f(b) = \{\theta(w) \mid [b]w \in P\}.$$

Hence, there is also a $[b]w \in P$ such that $\theta(v) = \theta(w)$. Since v, w satisfy the premise of cong_b , we can derive $[b]v \leftrightarrow [b]w$, which propositionally entails χ . $\square$

7.4.5 Monotone neighborhood logic

$$(\text{Mon}) \frac{v \rightarrow w}{[a]v \rightarrow [a]w}.$$

7.4.6 Conditional Logic

7.4.7 Graded Logic

7.4.8 Counterexample Logic?

7.4.9 Restriction functor abstraction Logic?

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