Friedrich-Alexander-Universität
Technische Fakultät

Implementing Categorical Notions of Partiality and Delay in Agda

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1. Partiality in Type Theory
2. Categorical Notions of Partiality
3. Implementation in Agda

Partiality in Haskell

In Haskell we are able to define arbitrary partial functions

- Some can be spotted easily by their definition:

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- others might be more subtle:

```haskell
reverse :: [a] -> [a]
reverse l = revAcc l []
  where
    revAcc []     a = a
    revAcc (x:xs) a = revAcc xs (x:a)
```

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reverse :: [a] -> [a]
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  where
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```

```
ghci> ones = 1 : ones
ghci> reverse ones
...
```

Partiality in Agda
The Maybe Monad

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- Simple errors can be modelled with the maybe monad

```
data Maybe (A : Set) : Set where
  just    : A → Maybe A
  nothing : Maybe A
```

```
head : ∀ A → List A → Maybe A
head nil          = nothing
head (cons x xs)  = just x
```

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head : ∀ A → List A → Maybe A
head nil           = nothing
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```

- What about `reverse` for (possibly) infinite lists:

```
data Colist (A : Set) : Set where
  [] : Colist A
  _∷_ : A → ∞ (Colist A) → Colist A
```

Partiality in Agda

Capretta's Delay Monad

- Capretta's Delay Monad is a **coinductive** data type whose inhabitants can be viewed as suspended computations.

```
data Delay (A : Set) : Set where
  now : A → Delay A
  later : ∞ (Delay A) → Delay A
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- The delay datatype contains a constant for non-termination:

```
never : Delay A
never = later (♯ never)
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- The delay datatype contains a constant for non-termination:

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```

- and we can define a function for *running* a computation (for some amount of steps):

```
run_for_steps : Delay A → ℕ → Delay A
run now   x for n      steps = now x
run later x for zero   steps = later x
run later x for suc n steps = run ♭ x for n steps
```

Partiality in Agda

Capretta's Delay Monad

- Now we can define a reverse function for possibly infinite lists:

```
reverse : ∀ {A : Set} → Colist A → Delay (Colist A)
reverse {A} l = revAcc l []
  where
    revAcc : Colist A → Colist A → Delay (Colist A)
    revAcc []        a = now a
    revAcc (x ∷ xs) a = later (♯ revAcc (♭ xs) (x ∷ (♯ a)))
```

Preliminaries

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- is distributive, i.e. the following is an iso:

$$(X \times Y) + (X \times Z) \xrightarrow{dstl^{-1} := [\langle id, inl \rangle, \langle id, inr \rangle]} X \times (Y + Z)$$

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- has a natural numbers object $\mathbb{N}$ (which is stable)

Capturing Partiality

Moggi's categorical semantics [Mog91]

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What properties should a monad T for modelling partiality have?

1. Commutativity (also entails strength)
2. Morphisms in $\mathcal{C}_T$ should be partial maps
3. There should be no other effect besides partiality

Capturing Partiality

Restriction Categories [CL02]

Definition

A restriction structure on $\mathcal{C}$ is a mapping $\text{dom} : \mathcal{C}(X, Y) \to \mathcal{C}(X, X)$ with the following properties:

$$f \circ (\text{dom } f) = f \tag{1}$$
$$(\text{dom } f) \circ (\text{dom } g) = (\text{dom } g) \circ (\text{dom } f) \tag{2}$$
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$$(\text{dom } h) \circ f = f \circ \text{dom } (h \circ f) \tag{4}$$

for any $X, Y, Z \in |\mathcal{C}|, f : X \to Y, g : X \to Z, h : Y \to Z$.

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Intuitively dom f captures the domain of definedness of f.

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for any $X, Y, Z \in |\mathcal{C}|, f : X \to Y, g : X \to Z, h : Y \to Z$.

Intuitively dom f captures the domain of definedness of f.

Remark

Every category has a trivial restriction structure dom $f = id$, we call categories with a non-trivial restriction structure *restriction categories*.

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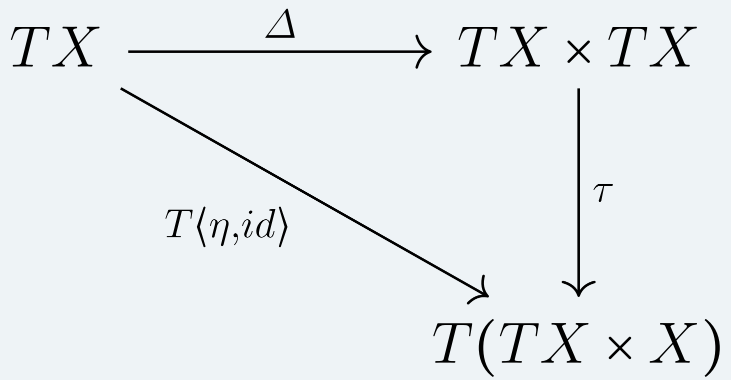
A commutative monad T is called an *equational lifting monad* if the following diagram commutes:



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A commutative monad T is called an *equational lifting monad* if the following diagram commutes:

$$\begin{array}{ccc} TX & \xrightarrow{\Delta} & TX \times TX \\ & \searrow^{T\langle \eta, id \rangle} & \downarrow{\tau} \\ & & T(TX \times X) \end{array}$$

Theorem

The Kleisli category of an equational lifting monad is a restriction category.

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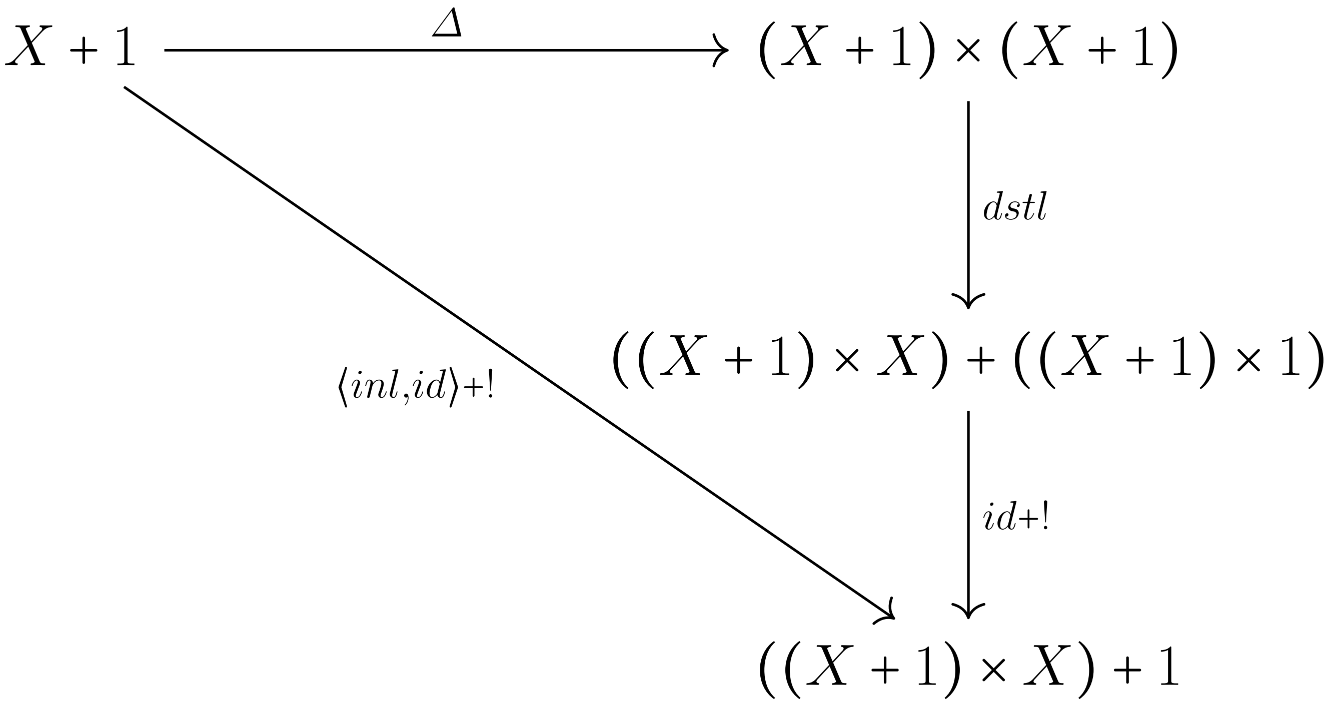

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Capretta's Delay Monad [Cap05]

- Recall the delay codatatype:

$$\frac{x : X}{\overline{\overline{\textit{now}\ x : DX}}} \qquad \frac{c : DX}{\overline{\overline{\textit{later}\ c : DX}}}$$

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- D (if it exists) is a strong and commutative monad
- D is not an equational lifting monad, because besides modelling partiality, it also counts steps (e.g. $now\ c \neq later\ (now\ c)$)

Capturing Partiality

Quotienting the Delay Monad [CUV15]

Following the work by Chapman, Uustalu and Veltri we can quotient D by the 'correct' kind of equality:

$$\frac{p \downarrow c \qquad q \downarrow c}{p \approx q} \qquad\qquad \frac{p \approx q}{later\, p \approx later\, q}$$

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we can model this as the coequalizer:

$$D(X \times \mathbb{N}) \underset{D fst}{\overset{\iota^*}{\rightrightarrows}} DX \xrightarrow{\rho_X} D_{\approx}X$$

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Problem: Defining $\mu_X : D^2_{\approx}X \to D_{\approx}X$ requires countable choice.

Partiality from Iteration
Elgot Algebras

The following is an adaptation of Adámek, Milius and Velebil's *complete Elgot Algebras* [AMV06]:

Definition

A (unguarded) Elgot Algebra [Gon21] consists of:

- An object X
- for every $f : S \to X + S$ the iteration $f^{\#} : S \to X$, satisfying:
  - **Fixpoint**: $f^{\#} = [id, f^{\#}] \circ f$
  - **Uniformity**: $(id + h) \circ f = g \circ h \Rightarrow f^{\#} = g^{\#} \circ h$
    for $f : S \to X + S,\ g : R \to A + R,\ h : S \to R$
  - **Folding**: $((f^{\#} + id) \circ h)^{\#} = [(id + inl) \circ f, inr \circ h]^{\#}$
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Remark

Every Elgot algebra $(A, (-)^\#)$ comes with a divergence constant $\bot = (inr : 1 \to A + 1)^\# : 1 \to A$

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A monad $\mathbf{T}$ is an Elgot monad if it has an iteration operator $(f : X \to T(Y + X))^\dagger : X \to TY$ satisfying:

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  for $f : X \to T(Y + X)$
- **Uniformity**: $f \circ h = T(id + h) \circ g \Rightarrow f^\dagger \circ h = g^\dagger$
  for $f : X \to T(Y + X), g : Z \to T(Y + Z), h : Z \to X$
- **Naturality**: $g^* \circ f^\dagger = ([(T inl) \circ g, \eta \circ inr]^* \circ f)^\dagger$
  for $f : X \to T(Y + X), g : Y \to TZ$
- **Codiagonal**: $f^{\dagger\dagger} = (T[id, inr] \circ f)^\dagger$
  for $f : X \to T((Y + X) + X)$

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Remark

Strong Elgot Monads are regarded as minimal semantic structures for interpreting effectful while-languages.

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pre-Elgot Monads [Gon21]

Relaxing the requirements for Elgot monads we get the following weaker concept:

Definition

A monad $\mathbf{T}$ is called pre-Elgot if every TX extends to an Elgot algebra such that Kleisli lifting is iteration preversing, i.e.

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Theorem

Every Elgot monad is pre-Elgot

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- By defining KX as the free Elgot algebra over X we get a monad K (that we assume is stable)

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Closing the gap [Gon21]

Let's look at $\mathbf{K}$ under various assumptions:

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- **Assuming countable choice:**
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2. Categorical Notions of Partiality
3. **Implementation in Agda**

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  - strong

Goals

- Formalize the delay monad categorically and show that it is..
  - strong
  - commutative
- Formalize K and show that it is..
  - strong
  - commutative

Goals

- Formalize the delay monad categorically and show that it is..
  - strong
  - commutative
- Formalize K and show that it is..
  - strong
  - commutative
  - an equational lifting monad

Goals

- Formalize the delay monad categorically and show that it is..
  - strong
  - commutative
- Formalize K and show that it is..
  - strong
  - commutative
  - an equational lifting monad
  - the initial pre-Elgot monad

Goals

- Formalize the delay monad categorically and show that it is..
  - strong
  - commutative
- Formalize K and show that it is..
  - strong
  - commutative
  - an equational lifting monad
  - the initial pre-Elgot monad
- Take the category of setoids and show that K instantiates to $D_{\approx}$

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1. Partiality in Type Theory
2. Categorical Notions of Partiality
3. Implementation in Agda

App: Category Theory in Agda

Setoid-enriched Categories

```
record Category (o ℓ e : Level) : Set (suc (o ⊔ ℓ ⊔ e)) where
  field
    Obj : Set o
    _⇒_ : Obj → Obj → Set ℓ
    _≈_ : ∀ {A B} → (A ⇒ B) → (A ⇒ B) → Set e

    id  : ∀ {A} → (A ⇒ A)
    _∘_ : ∀ {A B C} → (B ⇒ C) → (A ⇒ B) → (A ⇒ C)

  field
    assoc     : ∀ {A B C D} {f : A ⇒ B} {g : B ⇒ C} {h : C ⇒ D}
      → (h ∘ g) ∘ f ≈ h ∘ (g ∘ f)
    identityˡ : ∀ {A B} {f : A ⇒ B} → id ∘ f ≈ f
    identityʳ : ∀ {A B} {f : A ⇒ B} → f ∘ id ≈ f
    equiv     : ∀ {A B} → IsEquivalence (_≈_ {A} {B})
    ∘-resp-≈  : ∀ {A B C} {f h : B ⇒ C} {g i : A ⇒ B} → f ≈ h → g ≈ i → f ∘ g ≈ h ∘ i
```